Journal Homepage: - www.journalijar.com
INTERNATIONAL JOURNAL OF ADVANCED RESEARCH (IJAR)
ISSN NO. 2320-5407

Article DOI: 10.21474/IJAR01/7623
DOI URL: http://dx.doi.org/10.21474/IJAR01/7623

RESEARCH ARTICLE

APPLICATIONS OF GENERALIZED INVERSES BOTH IN HOMOGENEOUS AND NON-HOMOGENEOUS SYSTEM OF LINEAR EQUATIONS.

Mohammad Maruf Ahmed.
Assistant Professor, Department of Mathematics and Natural Sciences, BRAC University, 66 Mohakhali Dhaka – 1212, Bangladesh.

Manuscript Info

Manuscript History

Received: 23 June 2018
Final Accepted: 25 July 2018
Published: August 2018

Keywords:-
consistent, hermitian, row-echelon, homogeneous.

Abstract

This paper deals with applications of generalized inverses (g-inverses) both in homogeneous and non-homogeneous system of linear equations. General as well as particular solutions have been derived associated with g-inverses. An example is shown which gives the general solutions of a consistent system of linear equation through the help of g-inverse.

Introduction:-

When the ordinary inverse of a square matrix cannot be used for the solution of the linear equations, we can use the g-inverse and M-P g-inverse of the singular or rectangular matrices for the solutions of the systems of linear equations.

Theorem 1. *A necessary and sufficient condition for the equation* $A^+B^+ = C$ *to have a solution is* $A^+(ACB)B^+ = C$ *in which case the general solution is*

$$X = Y - AA^+YB^+B + ACB \,, \quad \text{where } Y \text{ is arbitrary.}$$

Proof. $C = A^+XB^+ = A^+AA^+XB^+BB^+ = A^+ACBB^+$ (1)

Since $A^+XB^+ = C$.

Conversely, if (1) holds, then $X = ACB$ is a particular solution of $A^+XB^+ = C$

Now any expression of the form $X = Y - AA^+YB^+B$ satisfies $A^+XB^+ = 0$ beacause

$$A^+[Y - AA^+YB^+B]B^+ = A^+YB^+ - A^+AA^+YB^+BB^+$$
$$= A^+YB^+ - A^+YB^+$$
$$= 0.$$

The general solution of a non-homogeneous equation $A^+XB^+ = C$ is equal to the general solution of $A^+YB^+ = 0$ + a particular solution of $A^+XB^+ = C$. Hence the general solution of $A^+XB^+ = C$ is given by

Corresponding Author:- Mohammad Maruf Ahmed.
Address:- Assistant Professor, Department of Mathematics and Natural Sciences, *BRAC University*, 66 Mohakhali Dhaka – 1212, Bangladesh.

$X = Y - AA^{+}YB^{+}B + ACB$ where Y is arbitrary.

Theorem 2. *If* $P = \begin{bmatrix} A \\ B \end{bmatrix}$ *and* $P^{-1} = [X\ Y]$ *then* $AX = I$ *and* $B^{+}X = 0$ *have a unique common solution given by* $X = (I - BB^{+})(A - ABB^{+})^{+}$.

Proof.

$$AX = A(I - BB^{+})(A - ABB^{+})^{+}$$
$$= (A - ABB^{+})(A - ABB^{+})^{+}$$
$$= I$$

$$B^{+}X = B^{+}(I - BB^{+})(A - ABB^{+})^{+}$$
$$= (B^{+} - B^{+}BB^{+})(A - ABB^{+})^{+}$$
$$= (B^{+} - B^{+})(A - ABB^{+})^{+}$$
$$= 0.$$

Theorem 3. *If* $Q = [C\,D]$ *and* $Q^{-1} = \begin{bmatrix} Z \\ T \end{bmatrix}$ *then* $ZC = I$ *and* $ZD^{+} = 0$ *have a unique common solution given by* $Z = (C - D^{+}DC)^{+}(I - D^{+}D)$.

Proof.

$$ZC = (C - D^{+}DC)^{+}(I - D^{+}D)$$
$$= (C - D^{+}DC)^{+}(C - D^{+}DC)$$
$$= I$$

$$ZD^{+} = (C - D^{+}DC)^{+}(I - D^{+}D)D^{+}$$
$$= (C - D^{+}DC)^{+}(D^{+} - D^{+}DD^{+})$$
$$= (C - D^{+}DC)^{+}(D^{+} - D^{+})$$
$$= 0.$$

Theorem 4. *If* $AX = I$ *and* $BX = 0$ *then* $S(A - XB)$ *is hermitian iff* $X = AB^{+}$ *where* $P = \begin{bmatrix} A \\ B \end{bmatrix}$ *and* $P^{-1} = [S\ T]$..

Proof. First suppose that $S(A - XB)$ is hermitian, then

$$S(A - XB) = [S(A - XB)]^{*} = (A - XB)^{*}S^{*}$$

$$A(S(A - XB))B^{*} = A(A - XB)^{*}S^{*}B^{*} = A(A - XB)^{*}(BS)^{*}. \quad (2)$$

Now $PP^{-1} = \begin{bmatrix} A \\ B \end{bmatrix}[S\ \ T] = \begin{bmatrix} AS & AT \\ BS & BT \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$\therefore AS = BT = I \text{ and } AT = BS = 0, \text{ so, } (BS)^{*} = 0.$$

Thus *(2)* gives $A(S(A - XB))B^{*} = 0$

or, $AS(A - XB))B^{*} = 0$

or, $I(A-XB)B^{*} = 0$

or, $(A-XB)B^{*} = 0$

or, $AB^{*} - XBB^{*} = 0$

or, $XBB^{*} = AB^{*}$.

Let BB^{*} be non-singular, then ($(BB^{*})^{-1}$ exists and

$$X = AB^{*}(BB^{*})^{-1} = AB^{+} = AB^{+}, \text{ since } B^{*}(BB^{*})^{-1} = B^{+}.$$

Conversely, if $X = AB^{+}$ then $S(A-XB)$ is hermitian.

Because $S(A-XB) = S(A-AB^{+}B)$ and $AS = I$ and $BS = 0$ have a unique common solution given by

$$S = (I-B^{+}B)(A-AB^{+}B)^{+}$$

$$\therefore \quad S(A-XB) = (I-B^{+}B)(A-AB^{+}B)^{+}(A-AB^{+}B)$$

$$[S(A-XB)]^{*} = [(I-B^{+}B)(A-AB^{+}B)^{+}(A-AB^{+}B)]^{*}$$

$$= [(A-AB^{+}B)^{*}[(A-AB^{+}B)^{+}]^{*}(I-B^{+}B)^{*}$$

$$= [(A-AB^{+}B)^{+}(A-AB^{+}B)]^{*}(I-B^{+}B)$$

since ($I-B^{+}B)$ is hermitian.

$$= (A-AB^{+}B)^{+}(A-AB^{+}B)(I-B^{+}B).$$

Theorem 5. *If* $ZC = I$ *and* $ZD = 0$ *then* $(C-DZ)L$ *is hermitian iff* $Z = D^{+}C$ *where* $Q = [C \; D]$ *and* $Q^{-1} = \begin{bmatrix} L \\ M \end{bmatrix}$.

Proof. First suppose that $(C-DZ)L$ is hermitian then

$$(C-DZ)L = [(C-DZ)L\}]^{*} = L^{*}(C-DZ)^{*}$$

$$D^{*}((C-DZ)L)C = D^{*}L^{*}(C-DZ)^{*}C = (LD)^{*}(C-DZ)^{*}C. \quad (3)$$

Now $Q^{-1}Q = \begin{bmatrix} L \\ M \end{bmatrix}[CD] = \begin{bmatrix} LC & LD \\ MC & MD \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\therefore LC = MD = I$ and $LD = MC = 0$. So, $(LD)^{*} = 0$.

Thus (3) gives $D^{*}((C-DZ)L)C = 0$

or, $D^{*}(C-DZ)LC = 0$

or , $D^{*}(C-DZ)I = 0$

or, $D^{*}(C-DZ) = 0$

or, $D^{*}C - D^{*}DZ = 0$

or, $D^{*}DZ = D^{*}C$.

Let, $D^{*}D$ be non-singular then ($D^{*}D)^{-1}$ exists and

$$Z=(D^*D)^{-1}D^*C=D^+C \text{ since } (D^*D)^{-1}D^* = D^+$$

Conversely, if $Z = D^+C$ then $(C-DZ)L$ is hermitian.

Because $(C-DZ)L = (C-DD^+C)L$ and

$LC = I$ and $LD = 0$ have a unique common solution given by

$$L = (C-DD^+C)^+(I-DD^+)$$

$$\therefore\ (C-DZ)L = (C-DD^+C)(C-DD^+C)^+(I-DD^+).$$

Also

$$[(C-DZ)L]^* = [(C-DD^+C)(C-DD^+C)^+(I-DD^+)]^*$$

$$= [(I-DD^+)^*[(C-DD^+C)^+]^*(C-DD^+C)^*$$

since ($I-DD^+$) is hermitian.

$$= [(I-DD^+)(C-DD^+C)^+(C-DD^+C)^* .$$

Theorem 6. *Consistent equations* $Ax = y$ *have a solution* $x = A^-y$ *iff* $AA^-A = A.$

Proof. If $Ax = y$ are consistent and have $x = A^-y$ as a solution, write a_i for the i-th column of A and consider the equations $Ax = a_i$. They have a solution, the null vector with its i-th element set equal to unity. Therefore, the equations $Ax = a_i$ are consistent.

Furthermore, since consistent equations $Ax = y$ have a solution $x = A^-y$, it follows that consistent equations $Ax = a_i$ have a solution $x = A^-a_i$ and this is true for all values of *i*, i.e , for all columns of A. Hence $AA^-A = A.$

Conversely, if $AA^-A = A$, then $AA^-Ax = Ax$ and when $Ax = y$ this gives $AA^-y = y$,

i.e $A(A^-y) = y.$

Hence $x = A^-y$ is a solution of $Ax = y$ and the theorem is proved.

Theorem 7. *If* A *has* q *columns and if* A^- *is a generalized inverse of* A, *then the consistent equations* $Ax = y$ *have the solution* $\bar{x} = A^-y + (A^-A - I)z,$ *, where z is an arbitrary vector of order* q.

Proof. We know

$$A\bar{x} = AA^-y + (AA^-A - A)z$$

$$= AA^-y, \text{ since } AA^-A = A$$

$$= y \ ; \text{ since } AA^-y = y$$

i.e. $\bar{x}$ satisfies $Ax = y$ and hence is solution.

Example. We have to find a particular solution and also the general solution of the following system of linear equations by using generalized inverse:

$$
\begin{aligned}
2x_1 + 3x_2 + x_3 + 3x_4 &= 14 \\
x_1 + x_2 + x_3 + 2x_4 &= 6 \\
3x_1 + 5x_2 + x_3 + 4x_4 &= 22.
\end{aligned}
$$

Solution. The given system of linear equations can be written in matrix-form as

$$
\begin{pmatrix} 2 & 3 & 1 & 3 \\ 1 & 1 & 1 & 2 \\ 3 & 5 & 1 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 14 \\ 6 \\ 22 \end{pmatrix}
$$

(4)

Let

$$
A = \begin{pmatrix} 2 & 3 & 1 & 3 \\ 1 & 1 & 1 & 2 \\ 3 & 5 & 1 & 4 \end{pmatrix}, \quad x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \text{ and } \quad y = \begin{pmatrix} 14 \\ 6 \\ 22 \end{pmatrix}
$$

The system (4) can be written as

$$
Ax = y. \tag{5}
$$

First, we will find out the generalized inverse of A for which we need the rank of A. Reduce the matrix A to row –echelon form by the elementary row operations.

$$
\begin{aligned}
A &= \begin{pmatrix} 2 & 3 & 1 & 3 \\ 1 & 1 & 1 & 2 \\ 3 & 5 & 1 & 4 \end{pmatrix} \\
&\sim \begin{pmatrix} 1 & 1 & 1 & 2 \\ 2 & 3 & 1 & 3 \\ 3 & 5 & 1 & 4 \end{pmatrix} \\
&\sim \begin{pmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & -1 & -1 \\ 0 & 2 & -2 & -2 \end{pmatrix} \\
&\sim \begin{pmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}
\end{aligned}
$$

This matrix is in row echelon form and has two non-zero rows. So, rank of A is 2. Now let us partition the matrix A in the following way:

$$A = \left(\begin{array}{cc|cc} 2 & 3 & 1 & 3 \\ 1 & 1 & 1 & 2 \\ \hline 3 & 5 & 1 & 4 \end{array}\right)$$

$$= \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \text{ where } A_{11} = \begin{pmatrix} 2 & 3 \\ 1 & 1 \end{pmatrix}$$

$$\text{Since, } |A_{11}| = \begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix} = 2-3 = -1 \neq 0$$

$$A_{11}^{-1} \text{ exists and } A_{11}^{-1} = \begin{pmatrix} -1 & 3 \\ 1 & -2 \end{pmatrix}.$$

Hence a g-inverse of A is

$$A^{-} = \begin{pmatrix} -1 & 3 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Thus a particular solution of the system is

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -1 & 3 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 14 \\ 6 \\ 22 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \\ 0 \\ 0 \end{pmatrix}.$$

$$\therefore\ x_1 = 4,\ x_2 = 2, x_3 = 0, x_4 = 0.$$

Now we will find out the general solution of the given system.

When A has q columns and A^{-} is a generalized inverse of A, then the consistent system $Ax = y$ have solutions $x = A^{-}y + (A^{-}A - I)z,$ where z is any arbitrary vector of order q.

$$A^{-}A - I = \begin{pmatrix} -1 & 3 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 2 & 3 & 1 & 3 \\ 1 & 1 & 1 & 2 \\ 3 & 5 & 1 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 2 & 3 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Therefore, the general solution of the linear system is

$$x = A^{-}y + (A^{-}A - I)z,$$

$$= \begin{pmatrix} 4 \\ 2 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 2 & 3 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 2z_3 + 3z_4 \\ -z_3 - z_4 \\ -z_3 \\ -z_4 \end{pmatrix}$$

or,

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 4+2z_3+3z_4 \\ 2-z_3-z_4 \\ -z_3 \\ -z_4 \end{pmatrix}$$

$$\left.\begin{aligned} x_1 &= 4 + 2z_3 + 3z_4 \\ x_2 &= 2 - z_3 - z_4 \\ x_3 &= -z_3 \\ x_4 &= -z_4 \end{aligned}\right\} \text{ for any values of } z_3 \text{ and } z_4.$$

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