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RESEARCH ARTICLE

EXISTENCE AND UNIQUENESS OF COMMON FIXED POINT OF WEAKLY COMPATIBLE MAPS IN G-METRIC SPACE.

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Abstract

In this paper we study concepts of weakly compatible maps and E.A. property of pair of self maps. We have used these concepts to prove common fixed point theorem of weakly compatible mappings in G-Metric space.
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Introduction:-

The fixed point Theory has various applications in different fields such as linear inequalities, Parameterize estimation problems. The first very important result of fixed point of contraction mapping was obtained by Stefan Banach [1] in 1922.

In 1976 Jungck [2] proved first common fixed point result for pair of commuting maps in ordinary metric space. The concept of weak commutativity which is a weaker type of commuting pair of maps was developed by Sesa [3] and proved some common fixed point theorems in metric space. In 1986 G. Jungck [4] derived the notion of compatible mappings to generalize the concept of weak commutative pair of maps .Then in 1986 Jungck [5] defined the concept of weakly compatible maps.And proved some common fixed point results. In 2002 , Aamri M.and D.E Moutawakil [6] introduced the concept of E.A. Property for pair of self maps.

In 1960 S. Gahler [7] derived a new structure of metric space called as 2-Metric Space and claimed that this is more generalized structure of Metric Space. But Some Authors Proved that there is no relation between these two mappings. Ha. etc. all [8] Stated that 2-metric need not be continuous of its variables, but ordinary metric is continuous of its variables.

In 1992 Bapurao Dhage [9] derived new generalized notion of metric space called as D-Metric Space .Mustafa Z. and Sims in 2003 [10] proved some of the results in D-metric Space are invalid.The concept of G-metric space was obtained by Mustafa and Sims [11] and proved several results of fixed point in G-metric Space. In 2012 Zead Mustafa [12] proved some results of common fixed points for weakly compatible maps.

Preliminaries:-

Definition 2.1 (G-Metric Space [11]). Let X be a non empty set and $G : X^3 \to R^+$ which satisfies the following conditions

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(1) $G(a,b,c)=0$ if $a=b=c$ i.e. for every a,b,c in X coincides.

(2) $G(a,a,b)>0$ for every $a,b,c \in X$ s.t. a $\neq b$.

(3)$G(a,a,b) \leq G(a,b,c)$ $\forall$ $a,b,c \in X$

(4)$G(a,b,c)=G(b,a,c)=G(c,b,a)=$.......(Symmetry in all three variables)

(5)$G(a,b,c) \leq G(a,x,x)+G(x,b,c)$, for all a,b,c,x in X (rectangle inequality)

Then the function G is said to be a generalized Metric Space or Simply G-Metric on X and the pair (X,G) is Said to be G-Metric Space.

Example 2.1 Let $G: X^3 \to R^+$ s.t. $G(a,b,c)=$ perimeter of the triangle with vertices at a,b,c in R^2, also by taking p in interior of the triangle then rectangle inequality is satisfied and the function G is a function on X.

Remark 2.1 G-Metric Space is the generalization of the ordinary metric Space that is every G-metric space (X,G) defines ordinary metric space (X, d_G)

$$d_G(a,b)=G(a,b,b)+G(a,a,b)$$

Example 2.2 Let (X,d) be the usual Metric space . Then the function $G: X^3 \to R^+$ s.t. $G(a,b,c)=$ $\max\{d(a,b),d(b,c),d(a,c)\}$ for all a,b,c in X is a G-Metric space.

Definition 2.2 A G-Metric space (X,G) is said to be symmetric if $G(a,b,b)=G(a,a,b)$ for all $a,b,c \in X$ and if $G(a,b,b) \neq G(a,a,b)$, then G is said to be non symmetric G-Metric space.

Example 2.3 Let X={x,y} and $G: X^3 \to R^+$defined by $G(x,x,x)=G(y,y,y)=0, G(x,x,y)=1$, $G(x,y,y)=2$ and extend G to all of X^3 by symmetry in the variables. Then X is a G-Metric space but it is non symmetric since G(x,x,y) $\neq$ G(x,y,y).

Definition 2.3 Let (X,G) be a G-Metric space , let $\{a_n\}$ be a sequence of elements in X. The sequence $\{a_n\}$ is said to be G-convergent to a if $\lim_{m,n\to\infty} G(a,a_n,a_m)=0$ i.e. for every ϵ >0 , there is N s.t. $G(a,a_n,a_m) < \epsilon$ for all m,n $\geq N$. It is denoted as $\lim_{n\to\infty} a_n = a$.

Proposition 2.1 ([11]) If (X,G) be a G-Metric space. Then following are equivalent.

(i) $\{a_n\}$ is G-convergent to a.

(ii) $G(a_n,a_n,a)$ $\to$ ∞ as n $\to \infty$

(iii) $G(a_n,a,a)$ $\to$ ∞ as n $\to \infty$

(iv) $G(a_m,a_n,a)$ $\to$ ∞ as m, n $\to \infty$

Definition 2.4 Let (X,G) be a G-Metric space. A sequence $\{a_n\}$ is called G-Cauchy if , for $\epsilon > 0$ there is an N $\in I^+$ (set of positive Integers) s.t.

$$G(a_n,a_m,a_l) < \epsilon \text{ for all n,m,l } \geq N$$

Proposition 2.2 Let (X,G) be a G-Metric space then the function G(a,b,c) is jointly continious in all three of its variables.

Proposition 2.3 ([11]) Let (X,G) be a G-Metric Space. Then for any a,b,c,x in X, it gives that

(i) If $G(a,b,c)=0$ then $a=b=c$

(ii) $G(a,b,c)\leq G(a,a,b)+G(a,a,c)$

(iii) $G(a,b,b)\leq 2G(b,a,a)$

(iv) $G(a,b,c)\leq G(a,x,c)+G(x,b,c)$

(v) $G(a,b,c)\leq \frac{2}{3}(G(a,x,x)+G(b,x,x)+G(c,x,x))$

Definition 2.5 If S and T be self maps of a set X .If w=Sx=Tx for some x in X, then x is called coincidence point of S and T.

Definition 2.6 [5] Self maps S and T are said to be weakly compatible if they commute at their conicidence point i.e. if Sx=Tx for some x in X then $STx=TSx$

Example 2.4 Let X=[1, +∞) and G(a,b,c)=|a-b|+|b-c|+|a-c|.

Define S,T : X $\rightarrow$ X by $S(a)=2a-1$ and $T(a)=a^2$, a ò X , we say that a=1 is the only coincidence point and S(T(1))=S(1)=1 and

T(S(1))=T(1)=1 , so S and T are weakly compatible.

Definition 2.7 [6] Let S and T be any two self maps on metric space (X,d). The pair of maps S and T are said to satisfy E.A. property if there exists a sequence $\{a_n\}$ in X s.t.

$$\lim_{n\to\infty} Sa_n = \lim_{n\to\infty} Ta_n = z \text{ , for some z in X.}$$

Example 2.5 Let X=[-1,1] and let G be the G-metric on X^3 to R^+ defined as follows

G(a,b,c)=|a-b|+|b-c|+|a-c|. Then (X,G) be a G-Metric Space.Let us define Sx=x and Tx= $\frac{x}{4}$.

Then for a sequence $a_n=\frac{1}{n}$. Then this gives $\lim_{n\to\infty} Sa_n = \lim_{n\to\infty} Ta_n = 0$,for 0 in X.

Here the pair of self maps satisfy E.A. property.

Manoj Kumar [13] Proved following result for pair of compatible maps.

Theorem 2.1 Let X be a complete G-Metric space. S,T:X $\rightarrow$ X be two compatible maps on X and which satisfies the following conditions.

(i) S(X) $\subseteq T$(X),

(ii) S or T is G-continuous,

(iii) G(Sa,Sb,Sc) ≤ q G(Ta,Tb,Tc) for every a,b,c in X and 0 ≤ q <1. And if S and T are compatible then S and T have unique common fixed point in X.

Also Latpate V.V. and Dolhare U.P. [14] Proved common fixed point theorem for the pair of compatible maps for the following contraction.

Theorem 2.2 Let X be a complete G-Metric space. S,T:X $\rightarrow$ X be two compatible maps on X and which satisfies the following conditions.

(i) S(X) $\subseteq T$(X),

(ii) S or T is G-continuous,

(iii) $G(Sa,Sb,Sc) \le \alpha G(Sa,Tb,Tc) + \beta G(Ta,Sb,Tc) + \gamma G(Ta,Tb,Sc) +$

$\delta G(Sa,Tb,Tc),$ *for* all a,b,c in X & α,β,γ and $\delta \ge 0$

s.t. $0 \le \alpha + 3\beta + 3\gamma + \delta < 1.$

Then S and T have unique common fixed point in X.

3 Main Result

Now we prove our main result for the pair of weakly compatible maps which satisfy (iii) in theorem 2.2.

Theorem 3.1:-Let (X,G) be a G-Metric Space which is Complete. If S and T be Weakly Compatible maps on X into itself , s.t.

(1) $S(X) \subseteq T(X)$

(2) $G(Sa,Sb,Sc) \le \alpha G(Sa,Tb,Tc) + \beta G(Ta,Sb,Tc) + \gamma G(Ta,Tb,Sc) +$

$\delta G(Sa,Tb,Tc),$ *for* all a,b,c in X & α,β,γ and $\delta \ge 0$

s.t. $0 \le \alpha + 3\beta + 3\gamma + \delta < 1$

(3)Subspace S(X) or T(X) is Complete.

Then there exists a Unique Common fixed point of S and T in X.

Proof:-Let us choose a_0 be an an arbitrary element in X. Since S(X) $\subseteq$ T(X), we design a sequence { b_n } in X s.t. for any a_1 in X $Sa_0 = Ta_1$. In general for a_{n+1} s.t.

$b_n = Sa_n = Ta_{n+1}$ for n=0,1,2...... From (2) in hypothesis, we have

$$G(Sa_n, Sa_{n+1}, Sa_{n+1}) \le \alpha G(Sa_n, Ta_{n+1}, Ta_{n+1}) + \beta G(Ta_n, Sa_{n+1}, Ta_{n+1})$$
$$+\gamma G(Ta_n, Ta_{n+1}, Sa_{n+1}) + \delta G(Sa_n, Ta_{n+1}, Ta_{n+1})$$

$\therefore$ from the above sequence ,we have

$$G(Sa_n, Sa_{n+1}, Sa_{n+1}) \le \beta G(Sa_{n-1}, Sa_{n+1}, Sa_n) + \gamma G(Sa_{n-1}, Sa_n, Sa_{n+1})$$
$$(\because \alpha G(Sa_n, Sa_n, Sa_n) = 0 = \delta G(Sa_n, Sa_n, Sa_n))$$

$\because$ By symmetry in variables, we have

$$G(Sa_{n-1}, Sa_{n+1}, Sa_n) = G(Sa_{n-1}, Sa_n, Sa_{n+1})$$
$$G(Sa_n, Sa_{n+1}, Sa_{n+1}) \le (\beta + \gamma) G(Sa_{n-1}, Sa_n, Sa_{n+1}) \quad (3.1)$$

By using rectangular inequality in G- metric space, We have

$$G(Sa_{n-1}, Sa_n, Sa_{n+1}) \le G(Sa_{n-1}, Sa_n, Sa_n) + G(Sa_n, Sa_{n+1}, Sa_n)$$
$$\le G(Sa_{n-1}, Sa_n, Sa_n) + 2G(Sa_n, Sa_{n+1}, Sa_{n+1})$$

($\because$ By using Proposition (2.3) (iii)) $\therefore$ from given hypothesis (ii), we have

$$(1 - 2\beta - 2\gamma) G(Sa_n, Sa_{n+1}, Sa_{n+1}) \le (\beta + \gamma) G(Sa_{n-1}, Sa_n, Sa_n)$$

$$G(Sa_n, Sa_{n+1}, Sa_{n+1}) \le \frac{\beta+\gamma}{1-2\beta-2\gamma} G(Sa_{n-1}, Sa_n, Sa_n)$$

$$G(Sa_n, Sa_{n+1}, Sa_{n+1}) \le q_1 G(Sa_{n-1}, Sa_n, Sa_n)$$

Where $q_1 = \frac{\beta+\gamma}{1-2\beta-2\gamma} < 1$

Continuing in this way, We get

$$G(Sa_n, Sa_{n+1}, Sa_{n+1}) \le q_1{}^n G(Sa_0, Sa_1, Sa_1) \tag{3.2}$$

For all n,m ò I^+ , Let m>n and by using rectangle inequality
Consider

$$G(b_n, b_m, b_m) \le G(b_n, b_{n+1}, b_{n+1}) + G(b_{n+1}, b_{n+2}, b_{n+2}) + \dots\dots + G(b_{m-1}, b_m, b_m)$$

$$G(b_n, b_m, b_m) \le (q_1^n + q_1^{n+1} + \dots\dots + q_1^{m-1}) G(b_0, b_1, b_1)$$

$(\therefore$ by using (3.2))

$$\le \frac{q_1^n}{1-q_1} G(b_0, b_1, b_1)$$

As $n, m \to \infty$ $\therefore$ R.H.S. of this inequality tends to 0. We have $lim_{n\to\infty} G(b_n, b_m, b_m) = 0$ $\therefore$ The sequence { b_n } is a G-Cauchy sequence in X. Since S(X) or T(X) is Complete subspace of X. Let T(X) is Complete then subsequence of { b_n } must get a limit in T(X).

$\therefore$ The Sequence { b_n } also convergent .Since { b_n } Contains a Convergent subsequence in T(X). Say it be c_1. Let $u = Tc_1^{-1}$ then Tu=c_1 Now we prove that Su=c_1

On putting $a = u, b = a_n$ and c=a_n in (ii), We have

$$G(Su, Sa_n, Sa_n) \le \alpha G(Su, Ta_n, Ta_n) + \beta G(Tu, Sa_n, Ta_n) + \gamma G(Tu, Ta_n, Sa_n) + \delta G(Su, Ta_n, Ta_n)$$

As $n \to \infty$,above inequality gives

$\beta G(Tu, Sa_n, Ta_n) = \beta G(c_1, c_1, c_1) = 0$ also

$\gamma G(Tu, Ta_n, Sa_n) = G(c_1, c_1, c_1) = 0$

$\therefore$ We have

$G(Su, c_1, c_1) \le (\alpha + \delta) G(Su, c_1, c_1)$

This gives , $Su = c_1$

$\therefore Su = Tu = c_1$ $\therefore$ u is a coincident point of S and T.

As S and T are weakly Compatible $\therefore$ By definition $STu = TSu$ $\therefore Sc_1 = Tc_1$

Now we prove that $Sc_1 = c_1$. If Possible suppose that $Sc_1 \ne c_1$,

$\therefore G(Sc_1,c_1,c_1) > 0$ In (ii) putting a=$c_1, b = u, c = u$

$\therefore$ We have

$$G(Sc_1,c_1,c_1) = G(Sc_1,Su,Su)$$
$$\leq \alpha G(Sc_1,Tu,Tu) + \beta G(Tc_1,Su,Tu)$$
$$+\gamma G(Tc_1,Tu,Su) + \delta G(Sc_1,Tu,Tu)$$
$$=(\alpha+\beta+\gamma+\delta) G(Sc_1,c_1,c_1)$$
$$< G(Sc_1,c_1,c_1)$$

This is a Contradiction. $\therefore$ which gives that $Sc_1 = c_1$

$\therefore Sc_1 = Tc_1 = c_1$ $\therefore$ that is c_1 is a Common fixed point of S and T.

For the Uniqueness,

IF Possible Suppose that c' is another Common fixed Point of S and T which is distinct from c_1. i.e. $c_1 \neq c'$.

Consider,

$$G(c_1,c',c') = G(Sc_1,Sc',Sc')$$
$$\leq \alpha G(Sc_1,Tc',Tc') + \beta G(Tc_1,Sc',Tc')$$
$$+\gamma G(Tc_1,Tc',Sc') + \delta G(Sc_1,Tc',Tc')$$
$$=(\alpha+\beta+\gamma+\delta) G(c_1,c',c')$$
$$< G(c_1,c',c')$$
$$\therefore c_1 = c'$$

Hence the proof.

Theorem 3.2 :-If S and T be two maps on a G-metric Space (X,G) into itself which Satisfy

$$(i)\ G(Sa,Sb,Sc) \leq \alpha G(Sa,Tb,Tc) + \beta G(Ta,Sb,Tc) + \gamma G(Ta,Tb,Sc) +$$
$$\delta G(Sa,Tb,Tc),\ for \text{ all a,b,c in X \& } \alpha,\beta,\gamma \text{ and } \delta \geq 0$$
$$\text{s.t. } 0 \leq \alpha + 3\beta + 3\gamma + \delta < 1$$

(ii) T(X) is closed subspace of X.

(iii) S and T satisfies E.A. property. Moreover, If S and T are weakly Compatible self Maps. Then S and T have Unique Common fixed Point in X.

Proof:-We have Given S and T satisfies E.A. Property $\therefore$ By definition , there exists a sequence $\{a_n\} \in X$ s.t. $\lim_{n\to\infty} Sa_n = \lim_{n\to\infty} Ta_n = z \in X$ Also by (ii) T(X) is closed, $\therefore$ every Convergent Sequence of Points of T(X) contains limit points.

$z \in T(X)$ $\therefore$ for some $y \in X$, z=Ty

$\therefore$ from (i) we have

$$G(Sy,Sa_n,Sa_n) \leq \alpha G(Sy,Ta_n,Ta_n) + \beta G(Ty,Sa_n,Ta_n)$$
$$+\gamma G(Ty,Ta_n,Sa_n) + \delta G(Sy,Ta_n,Ta_n)$$

As $n \to \infty$ and by $0 \leq \alpha + 3\beta + 3\gamma + \delta < 1$, Therefore which gives.

$$G(Sy,z,z) \le \alpha G(Sy,z,z) + \beta G(z,z,z)$$
$$+\gamma G(z,z,z) + \delta G(Sy,z,z)$$
$$=(\alpha+\delta)G(Sy,z,z)$$
$$\text{but } (\alpha+\delta) < 1$$
$$\therefore G(Sy,z,z) = 0$$
$$\therefore Sy = z$$
$$\therefore Sy = Ty = z \in X$$

$\therefore$ y is the Coincident point of S and T.
Also Given that S and T are weakly Compatible.

$\therefore$ Sz=STy=TSy=Tz

$$\therefore Sz = Tz$$

Again from (i) , We get

$$G(Sz,Sy,Sy) \le \alpha G(Sz,Ty,Ty) + \beta G(Tz,Sy,Ty)$$
$$+\gamma G(Tz,Ty,Sy) + \delta G(Sz,Ty,Ty)$$
$$\therefore G(Sz,z,z) \le \alpha G(Sz,z,z) + \beta G(Sz,z,z)$$
$$+\gamma G(Sz,z,z) + \delta G(Sz,z,z)$$
$$\le (\alpha+\beta+\gamma+\delta)G(Sz,z,z)$$
$$\therefore G(Sz,z,z) = 0$$
$$\therefore Sz = z$$
$$\therefore Sz = Tz = z$$

$\therefore$ z is a Common Fixed point of S and T.
Uniqueness easily followed from (i).

Conclusion:-

Thus we have proved Common fixed results for pair of weakly compatible maps and second result for weakly compatible maps which satisfy E.A. property.

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