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RESEARCH ARTICLE

A TWO DIMENSIONAL APPROACH FOR FINDING APPROXIMATED SOLUTIONS OF OPTIMIZATION PROBLEMS.

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Abstract

The problem of finding the minimum value of objective function appears in many fields such as mathematical programming, economics, engineering and others. In this paper, a two-dimensional approach for finding solutions of optimization problems is introduced. The proposed algorithm shows how to obtain a new point and a new direction in the feasible region, improving the process of solution depending on choosing three initial points from the feasible region without any conditions. The convergence of the proposed algorithm is discussed, and illustrative examples are presented.

Introduction:-

There are many methods to solve nonlinear programming problems such as cutting-plane method, feasible direction methods and others. Cutting-plane methods solve the optimization problems by approximating the feasible set or the objective function by a bundle of linear inequalities, called cutting planes. The approximation is iteratively refined by adding new cutting planes. Kelley's cutting plane method [4] was introduced in 1960 to solve nonlinear programming problems by solving a sequence of linear programming problems. In 2006 Claus Still [2] presented an algorithm that improves the convergence properties of the Extended Cutting Plane algorithm, taken into consideration the objective and constraint functions are continuously differentiable and convex. Also in 2010, he had presented another Sequential Cutting Plane algorithm extended to general continuously differentiable nonlinear programming problems containing both nonlinear inequality and equality constraints [3].

Other methods that solve a nonlinear programming problem are the feasible direction methods; these methods explore the feasible region by searching along directions which reduce the objective function while maintaining feasibility. A search direction is determined by using the initial point, the design is changed along this direction until either a minimum of the objective function is found or a constraint is encountered. At the new design, a new search direction is determined, and the design is then changed by moving along the new direction. For small moves in these directions, the designing must neither violate any constraint nor allow the objective function to increase.

Also, there are various ways of generating feasible directions and the most popular ones in terms of simplicity are Zoutendijk method [8], Rosen's gradient projection method [5], Wolfe's reduced gradient method [6], and Zangwill's convex simplex method [7].
In this paper, a two-dimensional method and an algorithm are presented. This method helps us to avoid some problems arising in the convergence of the above mentioned algorithms. The proposed algorithm is more efficient

than the above mentioned algorithms because the objective function and the constraints are not needed to be differentiable.

Finally, a complete comparison between the proposed method and Zoutendijk's method, Claus Still method is made by seeking solutions of some problems.

Problem formulation:-

Consider the following optimization problem:

$$P_1 \begin{cases} \min f(\mathrm{x}) \\ s.t \\ M = \left\{ X \in R^n : g_r(\mathrm{x}) \le 0, r = 1,2,...,m \right\}, \end{cases}$$

where $f(x), g_r(x), r = 1,2,....,\mathrm{m}$ are convex functions.

To find the solutions of the problem P_1, construct a sequence of sub problems P_k in two variables $(\alpha_1, \alpha_2) \in R^2$ based on the choice of $\mathrm{x}_k^1, \mathrm{x}_k^2, \mathrm{x}_k^3 \in M$ and calculate the corresponding values of the objective function at these points $f(\mathrm{x}_k^1), f(\mathrm{x}_k^2), f(\mathrm{x}_k^3)$ which have the order $f(\mathrm{x}_k^1) \le f(\mathrm{x}_k^3) \le f(\mathrm{x}_k^2)$ and form the function:

$\Phi(\alpha_1,\alpha_2) = f[\mathrm{x}_k^1 + \alpha_1(\mathrm{x}_k^2 - \mathrm{x}_k^1) + \alpha_2(\mathrm{x}_k^3 - \mathrm{x}_k^1)]$.

Consider

$N = \{(\alpha_1,\alpha_2) \in R^2 : \Psi_r(\alpha_1,\alpha_2) = g_r(\mathrm{x}_k^1 + \alpha_1(\mathrm{x}_k^2 - \mathrm{x}_k^1) + \alpha_2(\mathrm{x}_k^3 - \mathrm{x}_k^1)) \le 0, \mathrm{r} = 1,2,...,\mathrm{m}\}$.

Then find the point $(\bar{\alpha}_1, \bar{\alpha}_2) \in \mathrm{R}^2$ which is a solution of the sub problem:

$$P_k \begin{cases} \min \Phi(\alpha_1, \alpha_2) \\ s.t \\ N = \{(\alpha_1,\alpha_2) \in R^2 : \Psi_r(\alpha_1,\alpha_2) \le 0, \mathrm{r} = 1,2,...,\mathrm{m}\}. \end{cases}$$

One point of the three points chosen before that corresponding the maximum value of f is excluded and consider the new point $\bar{x}_{k+1} = x_k^1 + \bar{\alpha}_1(x_k^2 - x_k^1) + \bar{\alpha}_2(x_k^3 - x_k^1)$.

Main results:-

The basic idea of this study is based on constructing a two dimensional real valued function $\Phi(\alpha_1,\alpha_2)$ as:

$\Phi(\alpha_1,\alpha_2) = f(x_k^1 + \alpha_1(x_k^2 - x_k^1) + \alpha_2(x_k^3 - x_k^1))$ or $\Phi(\alpha_1,\alpha_2) = \mathrm{f}\left(x_k^1 + \alpha_1 p_1 + \alpha_1 p_2\right)$,

where x_k^1 is the point in the set M at which the least value of f and P_1, P_2 are the directions $(x_k^2 - x_k^1)$ and $(x_k^3 - x_k^1)$ respectively.

Lemma 3.1:- If $f : M \subset R^n \to R$ is a convex function on a convex set $M \subset R^n$ and for any $x_k^1, x_k^2, x_k^3 \in M$, the set $\tilde{N} = \left\{(\alpha_1,\alpha_2) \in R^2 : \Psi_r(\alpha_1,\alpha_2) = g_r(x_k^1 + \alpha_1(x_k^2 - x_k^1) + \alpha_2(x_k^3 - x_k^1) \le 0),\ r = 1,2,..,m\right\}$ is convex.

Proof

Let $(\bar{\alpha}_1, \bar{\alpha}_2), (\hat{\alpha}_1, \hat{\alpha}_2) \in \tilde{\mathrm{N}}$,

$g_r[x_k^1 + (\mu\bar{\alpha}_1 + (1-\mu)\hat{\alpha}_1)(x_k^2 - x_k^1) + (\mu\bar{\alpha}_2 + (1-\mu)\hat{\alpha}_2)(x_k^3 - x_k^1)] =$

$g_r[x_k^1 + \mu(\bar{\alpha}_1(x_k^2 - x_k^1) + \bar{\alpha}_2(x_k^3 - x_k^1)) + (1-\mu)(\hat{\alpha}_1(x_k^2 - x_k^1) + \hat{\alpha}_2(x_k^3 - x_k^1))] =$

$g_r[\mu(x_k^1 + \bar{\alpha}_1(x_k^2 - x_k^1) + \bar{\alpha}_2(x_k^3 - x_k^1)) + (1-\mu)(x_k^1 + \hat{\alpha}_1(x_k^2 - x_k^1) + \hat{\alpha}_2(x_k^3 - x_k^1))] \le$

$[\mu g_r[x_k^1 + \bar{\alpha}_1(x_k^2 - x_k^1) + \bar{\alpha}_2(x_k^3 - x_k^1)] + (1-\mu)g_r[x_k^1 + \hat{\alpha}_1(x_k^2 - x_k^1) + \hat{\alpha}_2(x_k^3 - x_k^1)]] \le 0$.

Lemma 3.2:- If $f : M \subset R^n \to R$ is a convex function on a convex set $M \subset R^n$, then for any $x_k^1, x_k^2, x_k^3 \in M$, $\Phi(\alpha_1, \alpha_2) = f(x_k^1 + \alpha_1(x_k^2 - x_k^1) + \alpha_2(x_k^3 - x_k^1))$ is convex on $\tilde{N}$.

Proof

Let $\lambda_1, \lambda_2 \in \tilde{N}$, then from lemma 3.1, $\mu\lambda_1 + (1-\mu)\lambda_2 \in \tilde{N}, \mu \in [0,\ 1]$.

Thus

$\Phi(\mu\lambda_1 + (1-\mu)\lambda_2) = f[x_k^1 + (\mu\lambda_1 + (1-\mu)\lambda_2)(x_k^2 - x_k^1) + (\mu\lambda_1 + (1-\mu)\lambda_2)(x_k^3 - x_k^1)] =$

$f[x_k^1 + \mu\lambda_1((x_k^2 - x_k^1) + (x_k^3 - x_k^1)) + (1-\mu)\lambda_2((x_k^2 - x_k^1) + (x_k^3 - x_k^1))] =$

$f[\mu(x_k^1 + \lambda_1(x_k^2 - x_k^1) + \lambda_1(x_k^3 - x_k^1)) + (1-\mu)(x_k^1 + \lambda_2(x_k^2 - x_k^1) + \lambda_2(x_k^3 - x_k^1))] \le$

$\mu\Phi(\lambda_1) + (1-\mu)\Phi(\lambda_2)$.

(From convexity of f)

Lemma 3.3:- If $(\bar{\alpha}_1, \bar{\alpha}_2)$ is the minimal solution of the problem $\min_{(\alpha_1,\alpha_2)\in N} \Phi(\alpha_1, \alpha_2)$. Then $f(\bar{x}_{k+1}) \le f(\mathrm{x}_k^1)$.

Proof

Since $(\bar{\alpha}_1, \bar{\alpha}_2)$ is the minimal solution of the problem $\min_{(\alpha_1,\alpha_2)\in N} \Phi(\alpha_1, \alpha_2)$, so

$\Phi(\bar{\alpha}_1, \bar{\alpha}_2) \le \Phi(\alpha_1, \alpha_2), \forall(\alpha_1, \alpha_2) \in N$.

Thus

$f(x_k^1 + \bar{\alpha}_1(x_k^2 - x_k^1) + \bar{\alpha}_2(x_k^3 - x_k^1)) \le f(x_k^1 + \alpha_1(x_k^2 - x_k^1) + \alpha_2(x_k^3 - x_k^1)), \forall(\alpha_1, \alpha_2) \in N$.

Since $(0,0) \in N$, $f(x_k^1 + \bar{\alpha}_1(x_k^2 - x_k^1) + \bar{\alpha}_2(x_k^3 - x_k^1)) \le f(x_k^1)$.

If $\bar{x}_{k+1} = x_k^1 + \bar{\alpha}_1(x_k^2 - x_k^1) + \bar{\alpha}_2(x_k^3 - x_k^1)$, then we get $f(\bar{x}_{k+1}) \le f(x_k^i), \mathrm{i} = 1, 2, 3$.

Note that lemma 3.3 means $\bar{\alpha}_1 P_1 + \bar{\alpha}_2 P_2$ is the descent direction of f on M.

Proposition3.4:- If $\bar{\alpha}_1 = \bar{\alpha}_2 = 0$ and $(\bar{\alpha}_1, \bar{\alpha}_2) \in N$ is the solution for P_k then $f(\bar{x}_{k+1}) = f(\bar{x}_k^1 + \bar{\alpha}_1(\bar{x}_k^2 - \bar{x}_k^1) + \bar{\alpha}_2(\bar{x}_k^3 - \bar{x}_k^1)) = f(\bar{x}_k^1),\ \forall\, \mathrm{k} \in R$.

Proof

Since $(\bar{\alpha}_1, \bar{\alpha}_2) \in R^2$ is the solution for P_k. So,

$f(\bar{x}_{k+1}) = f(\bar{x}_k^1 + \bar{\alpha}_1(\bar{x}_k^2 - \bar{x}_k^1) + \bar{\alpha}_2(\bar{x}_k^3 - \bar{x}_k^1)) = f(\bar{x}_k^1)$.

Since $(0,0) \in N$ then $f(\bar{x}_{k+1}) = f(\bar{\mathrm{x}}_k^1)$.

Lemma 3.5:- If $\overline{x}_{k+1} = \overline{x}_k^1 + \overline{\alpha}_k^1 \overline{P}_k^1 + \overline{\alpha}_k^2 \overline{P}_k^2 = \overline{x}_k^1$, then there is no other point $\hat{x} \in M$ such that $f(\hat{x}) \le f(\overline{x}_{k+1})$.

Proof

Assume $\hat{x} \in M$, $f(\hat{x}) \le f(\overline{x}_{k+1})$, and doesn`t lie in the direction $\overline{\alpha}_k^1 \overline{P}_k^1 + \overline{\alpha}_k^2 \overline{P}_k^2$.

Then there is another direction containing $\hat{x}$.

Since f is convex on M, $f(\lambda \hat{x} + (1-\lambda)\overline{x}_{k+1}) \le \lambda f(\hat{x}) + (1-\lambda) f(\overline{x}_{k+1})$.

Since $f(\hat{x}) \le f(\overline{x}_{k+1})$.

$f(\lambda \hat{x} + (1-\lambda)\overline{x}_{k+1}) \le f(\overline{x}_{k+1}) = f(\overline{x}_k^1 + \overline{\alpha}_k^1 \overline{P}_k^1 + \overline{\alpha}_k^2 \overline{P}_k^2)$, which contradict from lemma 3.3 that $\overline{\alpha}_1 P_1 + \overline{\alpha}_2 P_2$ is the descent direction of f on M. Hence there is no other point $\hat{x} \in M$ such that $f(\hat{x}) \le f(\overline{x}_{k+1})$.

Theorem 3.6:- The sequence generated by $x_{k+1} = x_k + \overline{\alpha}_k^1 P_1^k + \overline{\alpha}_k^2 P_2^k$ is convergent, when $\overline{\alpha}_k^1 + \overline{\alpha}_k^2$ tends to zero, where $(\overline{\alpha}_k^1, \overline{\alpha}_k^2)$ is the optimal solution of $\min\limits_{(\alpha_1,\alpha_2)\in N} \Phi(\alpha_1, \alpha_2)$.

Proof

Consider the ball B_k with center x_k and radius $\Delta = \max\left\{\left\|x_{k-1}^1 - x_k\right\|, \left\|x_{k-1}^3 - x_k\right\|\right\}$

$B_k(\overline{\alpha}_k^1, \overline{\alpha}_k^2) = \left\{x \in M : \left\|x - x_k\right\| \le (\overline{\alpha}_k^1 + \overline{\alpha}_k^2)\Delta\right\}$, which is closed and bounded.

The new point

$x_{k+1} = x_k + \overline{\alpha}_k^1 \overline{P}_k^1 + \overline{\alpha}_k^2 \overline{P}_k^2 \in B_k$, where $P_1^k = x_{k-1}^1 - x_k$, $P_2^k = x_{k-1}^3 - x_k$.

Since

$$\left\|x_{k+1} - x_k\right\| = \left\|x_k + \overline{\alpha}_k^1 \overline{P}_k^1 + \overline{\alpha}_k^2 \overline{P}_k^2 - x_k\right\| = \left\|\overline{\alpha}_k^1 \overline{P}_k^1 + \overline{\alpha}_k^2 \overline{P}_k^2\right\| \le \left\|\overline{\alpha}_k^1 \overline{P}_k^1\right\| + \left\|\overline{\alpha}_k^2 \overline{P}_k^2\right\|$$

$$\le \overline{\alpha}_k^1 \left\|\overline{P}_k^1\right\| + \overline{\alpha}_k^2 \left\|\overline{P}_k^2\right\| \le \overline{\alpha}_k^1 \Delta + \overline{\alpha}_k^2 \Delta \le (\overline{\alpha}_k^1 + \overline{\alpha}_k^2)\Delta \in B_k.$$

Similarly

$$x_{k+2} = x_{k+1} + \overline{\alpha}_{k+1}^1 \overline{P}_{k+1}^1 + \overline{\alpha}_{k+1}^2 \overline{P}_{k+1}^2 \in B_{k+1},$$

$$x_{k+3} = x_{k+2} + \overline{\alpha}_{k+2}^1 \overline{P}_{k+2}^1 + \overline{\alpha}_{k+2}^2 \overline{P}_{k+2}^2 \in B_{k+2}.$$

If $(\overline{\alpha}_k^1 + \overline{\alpha}_k^2)$ tends to zero, then from lemma 3.5, $x_{k+1} = x_k$ is a solution for the problem and the sequence $\{x_k\}$ is convergent to the solution.

The algorithm:-

From the previous discussion, The proposed algorithm proceeds as follows

Step 1:

Choose $I_1 = \{x_k^1, x_k^2, x_k^3\} \subset M$ then calculate $f(x_k^i)$, $i = 1, 2, 3$. In addition, suppose that $f(x_k^1) \le f(x_k^3) \le f(x_k^2)$, let $k = 0$.

Step 2:

Form the function:

$\Phi(\alpha_1, \alpha_2) = f[x_0^1 + \alpha_1(x_0^2 - x_0^1) + \alpha_2(x_0^3 - x_0^1)]$.

In addition, the functions:

$\Psi_r(\alpha_1, \alpha_2) = g_r[x_0^1 + \alpha_1(x_0^2 - x_0^1) + \alpha_2(x_0^3 - x_0^1)],\ r = 1, 2, ..., m$.

Step 3:
Solve the following problem:

$$\min \Phi(\alpha_1, \alpha_2)$$

$$s.t$$

$$N = \{(\alpha_1, \alpha_2) \in R^2 : \Psi_r(\alpha_1, \alpha_2) \le 0, \ \ r = 1, 2, ..., m\},$$

suppose that $(\bar{\alpha}_1, \bar{\alpha}_2)$ is the solution.

Step 4:
If $\bar{\alpha}_1 = \bar{\alpha}_2 = 0$ then go to step 5. Otherwise replace the point $x_0^2 \in I_1$ at which:

$f(x_0^2) \ge f(x_0^i), \ \ i = 1, 2, 3$.

And determine the point:

$\bar{x}_1 = x_0^1 + \bar{\alpha}_1(x_0^2 - x_0^1) + \bar{\alpha}_2(x_0^3 - x_0^1).$

Now we get $I_2 = \{\bar{x}_1, x_0^1, x_0^3\}$ then go to step 2.

Step 5:
At $\bar{\alpha}_1 = \bar{\alpha}_2 = 0$, then stop and the solution is $\bar{x}_{k+1} = x_k^1$.

Illustrative examples:-

Example 1

$$Minimize\{2x^2 + 2y^2 - 2xy - 4x - 6y\}$$

$$s.t$$

$$x + y \le 2,$$

$$x + 5y \le 5,$$

$$-x \le 0,$$

$$-y \le 0.$$

Table 1 The solution steps of Example 1 due to the proposed method.

K	I_k	f_k	α_k
1	(1,0.5) (1.5,0.25) (0.5,0.125)	-5.5 -3.625 -2.34375	$\bar{\alpha}_1 = -0.5419,$ $\bar{\alpha}_2 = -0.2838.$
Now calculate the point $\bar{x}_1 = (1.129, 0.7741)$, cancel the point $x_2 = (0.5, 0.125)$ to get $I_2 = \{(1.129, 0.7741), (1, 0.5), (1.5, 0.25)\}$.			
2	(1,0.5) (1.5,0.25) (1.129,0.7741)	-5.5 -3.625 -7.161	$\bar{\alpha}_1 = 0,$ $\bar{\alpha}_2 = 0.$
Now calculate the point $\bar{x}_2 = (1.129, 0.7741)$ that represents the solution.			

Example 2

$Minimize\{y\}$

$s.t$

$x^2 - y \le 0.$

Table 2 The solution steps of Example 2 due to the proposed method

K	I_k	f_k	α_k
1	(0.25,1) (1,3) (-0.5,2)	1 3 2	$\overline{\alpha}_1 = -0.444444444444433,$ $\overline{\alpha}_2 = -0.111111111111133.$
Now calculate the point $\overline{x}_1 = (0,0)$, cancel the point $x_2 = (1,3)$ to get $I_2 = \{(0,0), (0.25,1), (-0.5,2)\}$.			
2	(0,0) (-0.5,2) (0.25,1)	0 2 1	$\overline{\alpha}_1 = 0,$ $\overline{\alpha}_2 = 0.$
Now calculate the point $\overline{x}_2 = (0,0)$ that represents the solution.			

Example 3

$Minimize\{(x-2)^2 - (y-1)^2\}$

$s.t$

$x^2 - y \le 0.$

$y^2 - x \le 0.$

Table 3 The solution steps of Example 3 due to the proposed method

K	I_k	f_k	α_k
1	(0.2,0.1) (0.6,0.5) (0.5,0.5)	2.43 1.71 2	$\overline{\alpha}_1 = 1.00000000000094,$ $\overline{\alpha}_2 = -1.25000000000154.$
Now calculate the point $\overline{x}_1 = (1,1)$, cancel the point $x_2 = (0.5,0.5)$ to get $I_2 = \{(1,1), (0.2,0.1), (0.6,0.5)\}$.			
2	(0.2,0.1) (0.6,0.5) (1,1)	2.43 1.71 1	$\overline{\alpha}_1 = 0,$ $\overline{\alpha}_2 = 0.$
Now calculate the point $\overline{x}_2 = (1,1)$ that represents the solution.			

Example 4

$$Minimize\{x^2 + y^2 - 4x + 4\}$$

s.t

$$-x + y - 2 \le 0.$$

$$x^2 - y + 1 \le 0.$$

$$-x \le 0.$$

$$-y \le 0.$$

Table 4 The solution steps of Example 4 due to the proposed method

K	I_k	f_k	α_k
1	(0,1) (0.5,1.5) (0.7,2)	5 4.5 5.69	$\bar{\alpha}_1 = -0.823766166196313,$ $\bar{\alpha}_2 = -0.436654030913862.$
Now calculate the point $\bar{x}_1 = (0.5535737822, 1.306443932)$, cancel the point $x_2 = (0.7,\ 2)$ to get $I_2 = \{(0.5535737822, 1.306443932), (0,1), (0.5,1.5)\}$.			
2	(0,1) (0.5,1.5) (0.55357,1.30644)	5 4.5 3.798944550	$\bar{\alpha}_1 = 0,$ $\bar{\alpha}_2 = 0.$
Now calculate the point $\bar{x}_2 = (0.5535737822,\ 306443932)$ that represents the solution.			

Numerical results:-

This section is devoted to comparing between the proposed algorithm and the two algorithms for Zoutendijk [1] and SCP [2].

Table5 Comparison between the proposed algorithm and Zoutendijk algorithm.

Problem	Number of iterations of the proposed algorithm	Number of iterations of Zoutendijk algorithm
Example 1	2	3

Table 6 Comparison between the proposed algorithm and SCP algorithm.

Problem	Number of iterations of the proposed algorithm	Number of iterations of SCP algorithm
Example 2	2	9
Example 3	2	2
Example 4	2	3

The previous tables 5,6 shows that the number of iterations of the introduced algorithm is less than the number of iterations of Zoutendijk algorithm and SCP algorithm which means that the introduced algorithm is less in time, efforts and cost.

Conclusion:-

In this paper, a new algorithm for finding solutions of optimization problems is presented. The convergence of the proposed algorithm is discussed. A comparison with Zoutendijk and SCP has been done, which shows the efficiency of the proposed method. This algorithm saves time and efforts than some algorithms presented in literature such as Zoutendijk algorithm and SCP algorithm.

References:-

1. Bazaraa, M. S., Sherali, H. D. (2006): Nonlinear Programming Theory and Algorithm. 3rd Edition, Wiley, New York, pp. 543-548.
2. Claus Still., TapioWesterlund. (2006): A sequential cutting plane algorithm for solving convex NLP problems. European Journal of Operational Research, 173: 444–464.
3. Claus Still., TapioWesterlund. (2010): A linear programming-based optimization algorithm for solving nonlinear programming problems. European Journal of Operational Research, 200: 658–670.
4. Kelley, J.E., Jr. (1960): The cutting plane method for solving convex programs, Journal of the Society for Industrial and Applied Mathematics, 8 (4): 703–712.
5. Rosen, J. B. (1960): The Gradient Projection Method for Nonlinear Programming, Part I – Linear Constraints, SIAM Journal of Applied Mathematics, 8 (1): 181-217.
6. Wolfe, P. (1967): Methods of Nonlinear Programming, Nonlinear programming. North-Holland. Ed J. Abadie. 97-131.
7. Zangwill, W. I. (1967): The convex simplex method. Management Science, 14: 221-283.
8. Zoutendijk, G. (1960): Methods of Feasible Directions. Elsevier, Amsterdam and D Van Nostrand, Princeton, N.J.