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RESEARCH ARTICLE

ON APPROXIMATION BY STANCU TYPE OPERATORS IN MOBILE INTERVAL

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Abstract

In the present paper we propose Stancu type operators in mobile interval. We discuss its approximation properties. We also give an asymptotic estimate through Voronovskaja - type result for these operators.

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Introduction:-

Stancu [4] introduced an operator, known as Stancu operator,defined by

$$(S_n f)(x) = \sum_{k=0}^{n-sr} p_{n-sr,k}(x) \sum_{j=0}^{s} \left[p_{s,j}(x) f\left(\frac{k+jr}{n}\right)\right] \quad (1.1)$$

Where

$$p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}$$

here r and s are non-negative integer parameters satisfying the condition :$2sr < n$. For $r = 0$ or $s = 0$, Stancu operator is the Bernstein operator.Also when s=r=1, Stancu operators become the well known Bernstein operators.

For $e_j(t) = t^j$, j=0,1,2,

$$\begin{aligned}(S_n e_0)(x) &= 1\\ (S_n e_1)(x) &= x\\ (S_n e_2)(x) &= x^2 + \left(1 + \frac{sr(r-1)}{n}\right)\frac{x(1-x)}{n}\end{aligned}$$

M.A.Siddiqui et. al.[3] introduced in 2014, the class of new Bernstein type operators as

$$V_n^*(f;x) = \sum_{k=0}^{n} p_{n,k}(x) f\left(\frac{k}{n+1}\right) \quad (1.2)$$

where $f \in C\left[0, \frac{n}{n+1}\right]$ and

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$$p_{n,k}(x) = \left(\frac{1+n}{n}\right)^n \binom{n}{k} x^k \left(\frac{n}{n+1} - x\right)^{n-k} \qquad (1.3)$$

These operators were defined on mobile interval.

Now using the technique used in Stancu operators we propose the operator,

$$(S_n^* f)(x) = \sum_{k=0}^{n-sr} p_{n-sr,k}(x) \sum_{j=0}^{s} \left[p_{s,j}(x) f\left(\frac{k+jr}{n+1}\right)\right] \qquad (1.4)$$

where $f \in C\left[0, \frac{n}{n+1}\right]$, r and s are parameters which are non negative and satisfy the condition : $2sr < n$ and $p_{n,k}(x)$ is given by (1.3) . For r = 0 or s = 0 or s = r = 1 operators (1.4) reduce to operators (1.2).

Auxiliary Results:-

In this section we give some lemmas which will be used further in section 3 and 4.

Lemma2.1. For each $f \in C\left[0, \frac{n}{n+1}\right]$, $n \in N$ and non-negative integers r, s satisfying the condition: $2sr < n$ we have,

$$(i) S_n^*(e_0; x) = 1$$

$$(ii) S_n^*(e_1; x) = x$$

$$(iii) S_n^*(e_2; x) = x^2 + \left[1 + \frac{sr(r-1)}{n}\right]\frac{x}{n}\left(\frac{n}{n+1} - x\right)$$

$$(iv) S_n^*(e_3; x) = x^3\left[\frac{(n-1)(n-2)}{n^2} + \frac{2sr(r^2-1)}{n^3} - \frac{3sr(r-1)}{n^2}\right] + 3x^2\left[\frac{(n-1)}{n(n+1)} + \frac{sr(r-1)}{n(n+1)} - \frac{sr(r^2-1)}{n^2(n+1)}\right] + \frac{x}{(n+1)^2}\left[1 + \frac{sr(r^2-1)}{n}\right]$$

$$(v) S_n^*(e_4; x) = x^4\left[\frac{(n-1)(n-2)(n-3)}{n^3} + \frac{3(sr(r-1))^2}{n^4} - \frac{6sr(r^3-1)}{n^4} - \frac{6sr(r-1)}{n^2} + \frac{8sr(r^2-1)}{n^3}\right] + 6x^3\left[\frac{(n-1)(n-2)}{n^2(n+1)} - \frac{(sr(r-1))^2}{n^3(n+1)} + \frac{2sr(r^3-1)}{n^3(n+1)} + \frac{sr(r-1)}{n(n+1)} - \frac{2sr(r-1)}{n^2(n+1)} - \frac{2sr(r^2-1)}{n^2(n+1)}\right] + x^2\left[\frac{7(n-1)}{n(n+1)^2} + \frac{3(sr(r-1))^2}{n^2(n+1)^2} - \frac{7sr(r^3-1)}{n^2(n+1)^2} + \frac{6sr(r-1)}{n(n+1)^2} + \frac{4sr(r^2-1)}{n(n+1)^2}\right] + x\left[\frac{1}{(n+1)^3} + \frac{sr(r^3-1)}{n(n+1)^3}\right]$$

where $e_j(t) = t^j$, $j = 0, 1, 2, 3, 4$.

Proof. From (1.3) we get,

$$\sum_{k=0}^{s} p_{s,k}(x) = 1 \qquad (2.1)$$

$$\begin{aligned}
\sum_{k=0}^{s} p_{s,k}(x)\Big(\frac{k}{n+1}\Big) &= \sum_{k=0}^{s}\Big(\frac{1+n}{n}\Big)^{s}\binom{s}{k}x^{k}\Big(\frac{n}{n+1}-x\Big)^{s-k}\Big(\frac{k}{n+1}\Big)\\
&= \Big(\frac{1+n}{n}\Big)^{s}\sum_{k=1}^{s}\frac{s!}{(k-1)!(s-k)!}x^{k}\Big(\frac{n}{n+1}-x\Big)^{s-k}\Big(\frac{1}{n+1}\Big)\\
&= \frac{sx}{(n+1)}\Big(\frac{1+n}{n}\Big)^{s}\sum_{k=0}^{s-1}\frac{(s-1)!}{(k)!(s-1-k)!}x^{k}\Big(\frac{n}{n+1}-x\Big)^{s-1-k}\\
&= \frac{sx}{(n+1)}\Big(\frac{1+n}{n}\Big)^{s}\sum_{k=0}^{s-1}\binom{s-1}{k}x^{k}\Big(\frac{n}{n+1}-x\Big)^{s-1-k}\\
&= \frac{sx}{(n+1)}\Big(\frac{1+n}{n}\Big)^{s}\Big(\frac{n}{n+1}\Big)^{s-1}\\
&= \frac{sx}{n} \qquad (2.2)
\end{aligned}$$

and

$$\begin{aligned}
\sum_{k=0}^{s} p_{s,k}(x)\Big(\frac{k}{n+1}\Big)^{2} &= \sum_{k=0}^{s}\Big(\frac{1+n}{n}\Big)^{s}\binom{s}{k}x^{k}\Big(\frac{n}{n+1}-x\Big)^{s-k}\Big(\frac{k(k-1)+k}{(n+1)^{2}}\Big)\\
&= \frac{1}{(n+1)^{2}}\Big(\frac{1+n}{n}\Big)^{s}\Big[s(s-1)x^{2}\Big(\frac{n}{n+1}\Big)^{s-2} + sx\Big(\frac{n}{n+1}\Big)^{s-1}\Big]\\
&= s(s-1)\frac{x^{2}}{n^{2}} + \frac{sx}{n(n+1)} \qquad (2.3)
\end{aligned}$$

$$\begin{aligned}
\sum_{k=0}^{s} p_{s,k}(x)\Big(\frac{k}{n+1}\Big)^{3} &= \sum_{k=0}^{s}\Big(\frac{1+n}{n}\Big)^{s}\binom{s}{k}x^{k}\Big(\frac{n}{n+1}-x\Big)^{s-k}\Big(\frac{k(k-1)(k-2)}{(n+1)^{3}}\Big)\\
&= \sum_{k=0}^{s}\Big(\frac{1+n}{n}\Big)^{s}\binom{s}{k}x^{k}\Big(\frac{n}{n+1}-x\Big)^{s-k}\Big(\frac{3k(k-1)+k}{(n+1)^{3}}\Big)\\
&= \frac{1}{(n+1)^{3}}\Big(\frac{1+n}{n}\Big)^{s}\Big[s(s-1)(s-2)x^{3}\Big(\frac{n}{n+1}\Big)^{s-3}\\
&\quad + 3s(s-1)x^{2}\Big(\frac{n}{n+1}\Big)^{s-2} + sx\Big(\frac{n}{n+1}\Big)^{s-1}\Big]\\
&= s(s-1)(s-2)\frac{x^{3}}{n^{3}} + \frac{3s(s-1)x^{2}}{n^{2}(n+1)} + \frac{sx}{n(n+1)^{2}} \qquad (2.4)
\end{aligned}$$

$$\sum_{k=0}^{s} p_{s,k}(x)\Big(\frac{k}{n+1}\Big)^4 = \sum_{k=0}^{s}\Big(\frac{1+n}{n}\Big)^s\binom{s}{k}x^k\Big(\frac{n}{n+1}-x\Big)^{s-k}\Big(\frac{k(k-1)(k-2)(k-3)}{(n+1)^4}\Big)$$
$$+\sum_{k=0}^{s}\Big(\frac{1+n}{n}\Big)^s\binom{s}{k}x^k\Big(\frac{n}{n+1}-x\Big)^{s-k}\Big(\frac{6k(k-1)(k-2)}{(n+1)^4}\Big)$$
$$+\sum_{k=0}^{s}\Big(\frac{1+n}{n}\Big)^s\binom{s}{k}x^k\Big(\frac{n}{n+1}-x\Big)^{s-k}\Big(\frac{7k(k-1)+k}{(n+1)^4}\Big)$$
$$=\frac{1}{(n+1)^4}\Big(\frac{1+n}{n}\Big)^s\Big[s(s-1)(s-2)(s-3)x^4\Big(\frac{n}{n+1}\Big)^{s-4}$$
$$+6s(s-1)(s-2)x^3\Big(\frac{n}{n+1}\Big)^{s-3}+7s(s-1)x^2\Big(\frac{n}{n+1}\Big)^{s-2}$$
$$+sx\Big(\frac{n}{n+1}\Big)^{s-1}\Big]$$
$$=s(s-1)(s-2)(s-3)\frac{x^4}{n^4}+\frac{6s(s-1)(s-2)x^3}{n^3(n+1)}+\frac{7s(s-1)x^2}{n^2(n+1)^2}$$
$$+\frac{sx}{n(n+1)^3} \tag{2.5}$$

We have from (1.4),

$$(S_n^*f)(x)=\sum_{k=0}^{n-sr}p_{n-sr,k}(x)\sum_{j=0}^{s}\Big[p_{s,j}(x)f\Big(\frac{k+jr}{n+1}\Big)\Big]$$

Using (2.1), (2.2), (2.3), (2.4), (2.5) we compute,

$$S_n^*(e_0;x) = \sum_{k=0}^{n-sr}p_{n-sr,k}(x)\sum_{j=0}^{s}\Big[p_{s,j}(x)\Big]=1$$

$$S_n^*(e_1;x)=\sum_{k=0}^{n-sr}p_{n-sr,k}(x)\sum_{j=0}^{s}p_{s,j}(x)\Big(\frac{k+jr}{n+1}\Big)$$
$$=\sum_{k=0}^{n-sr}p_{n-sr,k}(x)\Big(\frac{k}{n+1}\Big)+r\sum_{j=0}^{s}p_{s,j}(x)\Big(\frac{j}{n+1}\Big)$$
$$=(n-sr)\frac{x}{n}+r\frac{sx}{n}$$
$$=x \tag{2.6}$$

$$S_n^*(e_2;x)=\sum_{k=0}^{n-sr}p_{n-sr,k}(x)\sum_{j=0}^{s}p_{s,j}(x)\Big(\frac{k+jr}{n+1}\Big)^2$$

$$= \sum_{k=0}^{n-sr} p_{n-sr,k}(x)\left(\frac{k}{n+1}\right)^2 + 2r\sum_{k=0}^{n-sr} p_{n-sr,k}(x)\left(\frac{k}{n+1}\right)\left[\sum_{j=0}^{s} p_{s,j}(x)\left(\frac{j}{n+1}\right)\right]$$

$$+ r^2\sum_{j=0}^{s} p_{s,j}(x)\left(\frac{j}{n+1}\right)^2$$

$$= (n-sr)(n-sr-1)\frac{x^2}{n^2} + (n-sr)\frac{x}{n(n+1)} + 2r\left[(n-sr)\frac{x}{n}\right]\left[\frac{sx}{n}\right]$$

$$+ r^2\left[s(s-1)\frac{x^2}{n^2} + \frac{sx}{n(n+1)}\right]$$

$$= x^2 + \left[1 + \frac{sr(r-1)}{n}\right]\frac{x}{n}\left(\frac{n}{n+1} - x\right) \qquad \mathbf{(2.7)}$$

Similarly,

$$S_n^*(e_3;x) = \sum_{k=0}^{n-sr} p_{n-sr,k}(x)\sum_{j=0}^{s} p_{s,j}(x)\left(\frac{k+jr}{n+1}\right)^3$$

$$= \sum_{k=0}^{n-sr} p_{n-sr,k}(x)\left(\frac{k}{n+1}\right)^3 + 3r\sum_{k=0}^{n-sr} p_{n-sr,k}(x)\left(\frac{k}{n+1}\right)^2\left[\sum_{j=0}^{s} p_{s,j}(x)\left(\frac{j}{n+1}\right)\right]$$

$$+ 3r^2\sum_{k=0}^{n-sr} p_{n-sr,k}(x)\left(\frac{k}{n+1}\right)\left[\sum_{j=0}^{s} p_{s,j}(x)\left(\frac{j}{n+1}\right)^2\right] + r^3\sum_{j=0}^{s} p_{s,j}(x)\left(\frac{j}{n+1}\right)^3$$

$$= x^3\left[\frac{(n-1)(n-2)}{n^2} + \frac{2sr(r^2-1)}{n^3} - \frac{3sr(r-1)}{n^2}\right] + 3x^2\left[\frac{(n-1)}{n(n+1)} + \frac{sr(r-1)}{n(n+1)}\right.$$

$$\left. - \frac{sr(r^2-1)}{n^2(n+1)}\right] + \frac{x}{(n+1)^2}\left[1 + \frac{sr(r^2-1)}{n}\right]$$

$$(2.8)$$

And

$$S_n^*(e_4;x) = \sum_{k=0}^{n-sr} p_{n-sr,k}(x)\sum_{j=0}^{s} p_{s,j}(x)\left(\frac{k+jr}{n+1}\right)^4$$

$$= \sum_{k=0}^{n-sr} p_{n-sr,k}(x)\left(\frac{k}{n+1}\right)^4 + 4r\sum_{k=0}^{n-sr} p_{n-sr,k}(x)\left(\frac{k}{n+1}\right)^3\left[\sum_{j=0}^{s} p_{s,j}(x)\left(\frac{j}{n+1}\right)\right]$$

$$+ 6r^2\sum_{k=0}^{n-sr} p_{n-sr,k}(x)\left(\frac{k}{n+1}\right)^2\left[\sum_{j=0}^{s} p_{s,j}(x)\left(\frac{j}{n+1}\right)^2\right] + 4r^3\sum_{k=0}^{n-sr} p_{n-sr,k}(x)$$

$$\times\left(\frac{k}{n+1}\right)\left[\sum_{j=0}^{s} p_{s,j}(x)\left(\frac{j}{n+1}\right)^3\right] + r^4\sum_{j=0}^{s} p_{s,j}(x)\left(\frac{j}{n+1}\right)^4$$

$$= x^4\Big[\frac{(n-1)(n-2)(n-3)}{n^3} + \frac{3(sr(r-1))^2}{n^4} - \frac{6sr(r^3-1)}{n^4} - \frac{6sr(r-1)}{n^2} + \frac{8sr(r^2-1)}{n^3}\Big] + 6x^3\Big[\frac{(n-1)(n-2)}{n^2(n+1)} - \frac{(sr(r-1))^2}{n^3(n+1)} + \frac{2sr(r^3-1)}{n^3(n+1)} + \frac{sr(r-1)}{n(n+1)} - \frac{2sr(r-1)}{n^2(n+1)} - \frac{2sr(r^2-1)}{n^2(n+1)}\Big] + x^2\Big[\frac{7(n-1)}{n(n+1)^2} + \frac{3(sr(r-1))^2}{n^2(n+1)^2} - \frac{7sr(r^3-1)}{n^2(n+1)^2} + \frac{6sr(r-1)}{n(n+1)^2} + \frac{4sr(r^2-1)}{n(n+1)^2}\Big] + x\Big[\frac{1}{(n+1)^3} + \frac{sr(r^3-1)}{n(n+1)^3}\Big] \tag{2.9}$$

□

Lemma2.2. For each $x \in \left[0, \frac{n}{n+1}\right]$, $n \in N$ and non- negative integers r,s satisfying the condition: $2sr < n$, following equalities hold,

$$(i) S_n^*(t-x;x) = 0$$

$$(ii) S_n^*((t-x)^2;x) = \Big[1 + \frac{sr(r-1)}{n}\Big]\frac{x}{n}\Big(\frac{n}{n+1} - x\Big)$$

$$(iii) S_n^*((t-x)^3;x) = \frac{2x^3}{n^2}\Big[1 + \frac{sr(r^2-1)x^3}{n}\Big] - \frac{3x^2}{n(n+1)}\Big[1 + \frac{sr(r^2-1)x^2}{n}\Big] + \frac{x}{(n+1)^2}\Big[1 + \frac{sr(r^2-1)x}{n}\Big]$$

$$(iv) S_n^*((t-x)^4;x) = \frac{x^4}{n^2}\Big[3 - \frac{6}{n} + \frac{3(sr(r-1))^2}{n^2} - \frac{6sr(r^3-1)}{n^2}\Big] - \frac{x^3}{n(n+1)}\Big[6 - \frac{12}{n} + \frac{6(sr(r-1))^2}{n^2} - \frac{12sr(r^3-1)}{n^2} + \frac{12sr(r-1)}{n}\Big] + \frac{x^2}{(n+1)^2}\Big[3 - \frac{7}{n} + \frac{6sr(r-1)}{n} + \frac{3(sr(r-1))^2}{n^2} - \frac{7sr(r^3-1)}{n^2}\Big] + \frac{x}{(n+1)^3}\Big[1 + \frac{sr(r^3-1)}{n}\Big] \tag{2.10}$$

Proof. From lemma 2.1 we get,

$$S_n^*(t-x;x) = S_n^*(t;x) - xS_n^*(1;x) = 0$$

$$\begin{aligned} S_n^*((t-x)^2;x) &= S_n^*(t^2;x) - x^2 - 2xS_n^*(t-x;x) \\ &= x^2 + \Big[1 + \frac{sr(r-1)}{n}\Big]\frac{x}{n}\Big(\frac{n}{n+1} - x\Big) - x^2 \\ &= \Big[1 + \frac{sr(r-1)}{n}\Big]\frac{x}{n}\Big(\frac{n}{n+1} - x\Big) \end{aligned}$$

$$S_n^*((t-x)^3;x) = S_n^*(t^3;x) - 3x^2S_n^*(t^2;x) + 3xS_n^*(t;x) - x^3$$

$$= x^3\Big[\frac{(n-1)(n-2)}{n^2} + \frac{2sr(r^2-1)}{n^3} - \frac{3sr(r-1)}{n^2}\Big] + 3x^2\Big[\frac{(n-1)}{n(n+1)} + \frac{sr(r-1)}{n(n+1)} - \frac{sr(r^2-1)}{n^2(n+1)}\Big] + \frac{x}{(n+1)^2}\Big[1 + \frac{sr(r^2-1)}{n}\Big]$$
$$- 3x\Big[x^2 + \Big(1 + \frac{sr(r-1)}{n}\Big)\frac{x}{n}\Big(\frac{n}{n+1} - x\Big)\Big] + 3x^3 - x^3$$
$$= \frac{2x^3}{n^2}\Big[1 + \frac{sr(r^2-1)x^3}{n}\Big] - \frac{3x^2}{n(n+1)}\Big[1 + \frac{sr(r^2-1)x^2}{n}\Big] + \frac{x}{(n+1)^2}\Big[1 + \frac{sr(r^2-1)x}{n}\Big]$$

Similarly, from lemma (2.1), we get,

$$S_n^*((t-x)^4;x) = S_n^*(t^4;x) - 4xS_n^*(t^3;x) + 6x^2S_n^*(t^2;x) - 4x^3S_n^*(t;x) + x^4$$
$$= \frac{x^4}{n^2}\Big[3 - \frac{6}{n} + \frac{3(sr(r-1))^2}{n^2} - \frac{6sr(r^3-1)}{n^2}\Big] - \frac{x^3}{n(n+1)}\Big[6 - \frac{12}{n} + \frac{6(sr(r-1))^2}{n^2} - \frac{12sr(r^3-1)}{n^2} + \frac{12sr(r-1)}{n}\Big] + \frac{x^2}{(n+1)^2}\Big[3 - \frac{7}{n} + \frac{6sr(r-1)}{n} + \frac{3(sr(r-1))^2}{n^2} - \frac{7sr(r^3-1)}{n^2}\Big] + \frac{x}{(n+1)^3}\Big[1 + \frac{sr(r^3-1)}{n}\Big]$$

□

Lemma2.3. For each $x \in \left[0, \frac{n}{n+1}\right]$, $n \in N$ and non-negative integers r,s satisfying the condition : $2sr < n$, following relations hold,

(i) $S_n^*((t-x)^2;x) < \frac{r+1}{8n}$

(ii) $\lim_{n\to\infty} nS_n^*((t-x)^2,x) = x(1-x)$

(iii) $\lim_{n\to\infty} n^2S_n^*((t-x)^4,x) = 3x^2(1-x)^2$

Proof. We know that maximum value of $x\left(\frac{n}{n+1} - x\right)$ in the interval $\left[0, \frac{n}{n+1}\right]$, $n \in N$ is $\frac{n^2}{4(n+1)^2}$.

Also r,s are non-negative integers satisfying the condition $2sr < n \Rightarrow sr < \frac{n}{2}$. So from lemma (2.2) we obtain,

$$S_n^*((t-x)^2;x) \le \left[1+\frac{sr(r-1)}{n}\right]\frac{1}{n}\left(\frac{n^2}{4(n+1)^2}\right)$$
$$\le \frac{1}{4n}\left[1+\frac{sr(r-1)}{n}\right]$$
$$< \frac{1}{4n}\left[1+\frac{(r-1)}{2}\right]$$
$$= \frac{r+1}{8n}$$

And,

$$nS_n^*((t-x)^2;x) = \left[1+\frac{sr(r-1)}{n}\right]x\left(\frac{n}{n+1}-x\right)$$
$$\lim_{n\to\infty} nS_n^*((t-x)^2;x) = x(1-x)$$

Again from lemma (2.2),

$$n^2S_n^*((t-x)^4,x) = x^4\left[3-\frac{6}{n}+\frac{3(sr(r-1))^2}{n^2}-\frac{6sr(r^3-1)}{n^2}\right]-6\frac{x^3n^2}{n(n+1)}\left[1-\frac{2}{n}\right.$$
$$\left.+\frac{(sr(r-1))^2}{n^2}-\frac{2sr(r^3-1)}{n^2}+\frac{2sr(r-1)}{n}\right]+\frac{x^2n^2}{(n+1)^2}\left[3-\frac{7}{n}\right.$$
$$\left.+\frac{6sr(r-1)}{n}+\frac{3(sr(r-1))^2}{n^2}-\frac{7sr(r^3-1)}{n^2}\right]$$
$$+\frac{xn^2}{(n+1)^3}\left[1+\frac{sr(r^3-1)}{n}\right]$$

Therefore,

$$\lim_{n\to\infty} n^2S_n^*((t-x)^4,x) = 3x^4-6x^3+3x^2 = 3x^2(1-x)^2$$

□

Main Result:-

In this section we give our main result regarding the convergence of the operator $S_n^* f$.

Theorem 3.1. If $f \in C\left[0,\frac{n}{n+1}\right]$, $n \in N$; r and s are parameters which are non negative and satisfy the condition : $2sr < n$, then $S_n^* f$ given by (1.4) converges uniformly to f on $\left[0,\frac{n}{n+1}\right]$.

Proof. From lemma (2.1) we see that,

$$\lim_{n\to\infty}(S_n^*e_j)(x) = x^j, \quad j=0,1,2,$$

where $e_j(t)=t^j$.

Hence by means of Bohman-Korovkin theorem [2], we obtain the desired result. □

A Voronovskaja-TypeTheorem

In this section we give aVoronovskaja-Type theorem for our operator.

Lemma4.1. Suppose that x_0 is a fixed point in $\left[0, \frac{n}{n+1}\right]$, $n \in N$ and $\phi(t;x_0)$ is a given bounded function belonging to $C\left[0, \frac{n}{n+1}\right]$, such that

$$\lim_{t \to x_0} \varphi(t; x_0) = 0$$

then,

$$\lim_{n \to \infty} S_n^*(\varphi(t; x_0); x_0) = 0. \quad (4.1)$$

where $S_n^* f$ is given by (1.4).

Proof. By (1.4) we have for $n \in N$ and a fixed point $x_0 \in \left[0, \frac{n}{n+1}\right]$,

$$S_n^*(\varphi(t; x_0); x_0) = \sum_{k=0}^{n-sr} p_{n-sr,k}(x) \sum_{j=0}^{s} \left[p_{s,j}(x) \varphi\left(\frac{k+jr}{n+1}, x_0\right) \right] \quad (4.2)$$

where r and s are parameters which are nonnegative and satisfy the condition : $2sr < n$.

Choose $\epsilon > 0$.Since $\varphi(\cdot; x_0) \in C[0, \frac{n}{n+1}]$, there exists a positive constant $\delta \equiv \delta(\epsilon)$ such that

$$|\varphi(t; x_0)| < \frac{\epsilon}{2} \ , \ if \ |t - x_0| < \delta, t \geq 0$$

Also, since $\varphi(\cdot; x_0)$ is bounded, therefore there exists a positive constant M such that $|\varphi(t; x_0)| \leq M$ for all $t > 0$. For $n > 0$ put,

$$A_n = \left\{ k \in N_0 : \left|\frac{k+jr}{n+1} - x_0\right| < \delta \right\}$$

Then from (4.2) we get for every $r, n, s \in N$

$$\begin{aligned} \left| S_n^*(\varphi(t; x_0); r; x_0) \right| &\leq \sum_{k \in A_n} p_{n-sr,k}(x) \sum_{j \in A_n} \left[p_{s,j}(x) \varphi\left(\frac{k+jr}{n+1}, x_0\right) \right] \\ &\quad + \sum_{k \notin A_n} p_{n-sr,k}(x) \sum_{j \notin A_n} \left[p_{s,j}(x) \varphi\left(\frac{k+jr}{n+1}, x_0\right) \right] \\ &< \frac{\epsilon}{2} + S_1 \end{aligned} \quad (4.3)$$

Now we have,

$$S_1 \leq M \sum_{k \notin A_n} p_{n-sr,k}(x) \sum_{j \notin A_n} \left[p_{s,j}(x) \varphi\left(\frac{k+jr}{n+1}, x_0\right) \right]$$

Since $|\frac{k+jr}{n+1} - x_0| \geq \delta$,using lemma (2.3) we can write,

$$S_1 \leq M\delta^{-2} \sum_{k \notin A_n} p_{n-sr,k}(x) \sum_{j \notin A_n} \left[p_{s,j}(x) \left(\frac{k+jr}{n+1} - x_0\right)^2 \right]$$

$$\leq M\delta^{-2}S_n^*((t-x_0)^2;x_0)$$
$$\leq M\delta^{-2}\left(\frac{r+1}{8n}\right)$$

It is obvious that for given $\epsilon > 0$, $\delta > 0$, $M > 0$ and $r \in N_0$ we can choose $n_0 \equiv n_0(\epsilon;\delta;M;r) \in N$ such that for all natural numbers $n > n_0$

$$M\delta^{-2}\left(\frac{r+1}{8n}\right) < \frac{\epsilon}{2}$$

Here

$$S_1 < \frac{\epsilon}{2} \quad , for\ n > n_0 \tag{4.4}$$

$$\lim_{n\to\infty} S_n^*(\varphi(t;x_0);x_0) = 0..$$

□

Theorem 4.2. Let $f^2 \in C\left[0,\frac{n}{n+1}\right]$, then

$$\lim_{n\to\infty} n\{S_n^*(f;x) - f(x)\} = \frac{1}{2}x(1-x)f''(x). \tag{4.5}$$

Proof. For a fixed $x \in [0,\frac{n}{n+1}]$,by Taylor's formula we can write for every $t \in [0,\frac{n}{n+1}]$,

$$f(t) = f(x) + f'(x)(t-x) + \frac{1}{2}f''(x)(t-x)^2 + \psi(t;x)(t-x)^2, \tag{4.6}$$

Then by assumption $\psi(t;x) \in C[0,\frac{n}{n+1}]$ is a bounded function and $\lim_{t\to x}\psi(t;x) = 0$
From this we have for every $n \in N$,

$$S_n^*(f;x) - f(x) = f'(x)S_n^*(t-x;x) + \frac{1}{2}f''(x)S_n^*((t-x)^2;x) + S_n^*(\psi(t;x)(t-x)^2;x) \tag{4.7}$$

and using lemma (2.2) we have

$$n[S_n^*(f;x) - f(x)] = \frac{1}{2}f''(x)[nS_n^*((t-x)^2;x)] + nS_n^*(\psi(t;x)(t-x)^2;x) \tag{4.8}$$

By Cauchy-Schwarz inequality we get for $n \in N$

$$|nS_n^*(\psi(t;x)(t-x)^2;x)| \leq \{S_n^*(\psi^2(t;x);x)\}^{\frac{1}{2}}\{n^2S_n^*((t-x)^4;x)\}^{\frac{1}{2}} \tag{4.9}$$

Since $\lim_{t\to x}\psi(t;x) = 0$,therefore, $\lim_{t\to x}\psi^2(t;x) = 0$.Hence by lemma (4.1) we get,

$$\lim_{n\to\infty} S_n^*(\psi^2(t;x);x) = 0. \tag{4.10}$$

Hence by lemma (2.2) and eqs. (4.8),(4.10) we conclude,

$$\lim_{n\to\infty} n\{S_n^*(f;x) - f(x)\} = \frac{1}{2}x(1-x)f''(x).$$

□

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