ISSN NO. 2320-5407

Journal homepage: http://www.journalijar.com

INTERNATIONAL JOURNAL OF ADVANCED RESEARCH

RESEARCH ARTICLE

On the utility of invariants for the solutions of variations

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Manuscript Info

Manuscript History:

Received: 15 May 2015
Final Accepted: 22 June 2015
Published Online: July 2015

Key words:

Constructivist approach, Chemistry Achievement, secondary school, Co-educational, Kenya

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Abstract

The purpose of this study was to determine the effects of Constructivist Teaching Approach (CTA) on students' achievement in Chemistry. The study was Quasi-experimental research and Solomon-Four Non-Equivalent Control Group Research Design. The target population comprised of Form Two Students in Baringo North Sub-County secondary schools. The accessible population was Form Two students in the Sub-County co-educational secondary schools. Purposive sampling was used to obtain a sample of four Co-educational Secondary Schools. Each school provided one Form Two class for the study hence a sample size of 160 students. The students were taught the same Chemistry topic of "Structure and Bonding". In the experimental group, constructivist teaching approach was used while the conventional teaching method was used in the control groups. The experimental groups were exposed to the Constructivist Teaching Approach (CTA) for a period of three weeks. The researcher trained the Chemistry teachers in the experimental groups on the technique of CTA before the treatment. The instrument used in the study was Chemistry Achievement Test (CAT) to measure students' achievement. Pilot test was done in a school in a different Division from the ones under study to ascertain the reliability of the instruments. Experts ascertained the validity of instruments before being used for data collection. The reliability coefficient α was 0.78. Data was analyzed using t-test, ANOVA and ANCOVA. Hypothesis was accepted or rejected at significance level of 0.05. The results of the study show that the CTA resulted in significantly higher students' achievement in Chemistry. The results of this study may be beneficial to Chemistry teachers, teacher trainers and curriculum developers in improving the teaching-learning process and achievement in Chemistry.

INTRODUCTION

Invariance has proved to be very useful in solving variations and we can say that one of the key ideas in mathematical problem solving is exploiting invariance. They always simplify matters**.**

The concept of an invariant is quite complex. Throughout these years, the word has taken many different meanings and has been applied to many different objects. From our point of view, invariants are objects that remain in a way unchanged under some transformation. However, in general, the concept of transformation is not always present in the definition of invariants. For example, the dimension of a variety is sometimes called an invariant of the variety. While it is clear that this quantity does reveal something essential about the variety, **no** transformation is involved in that case.

A more correct way to define invariants substitutes the concept of transformation by the more general concept of equivalence relation. In this broader context, an invariant is something that is common to all the objects belonging to the same equivalence class.

In general, an invariant is defined as real valued function that is unaffected by a group transformations. The determination of a complete set of invariants of a given group action is a problem of supreme importance for the study of equivalence and canonical forms, for a group action are completely characterized by its invariants.

2. Some preliminaries

2.1 Definition

We say that a real valued function $I: U \subseteq M \longrightarrow \mathrm{R}$ is an *invariant* if $I(g \,.\, x) = I(x)$, for all $g \in G$ and all $x \in U$, we say that a real valued function $I: U \subseteq M \longrightarrow \mathrm{R}$ is a *local invariant* if there exists a neighborhood N of the identity $e \in G$ such that $I(g \,.\, x) = I(x)$, for all $g \in N$ and all $x \in U$.

For instance, if $G = SO(2)$ acts on R^2, then the distance from a point to the origin is a global invariant of the group action.

In other words, invariants are quantities that remain constant on the orbits of a group action. Therefore the concept of orbit is closely related to the concept of invariant.

2.2 Proposition

Let $I : X \longrightarrow R$. The following conditions are equivalent

(a) I is a G-invariant function.

(b) I is constant on the orbits of G.

(c) All level sets { I(x) = c } are G-invariants subsets of X.

In particular, constant functions are trivially G-invariant. If G acts transitively on X ,then these are the only invariants.

On a smooth manifold, we will focus mostly on the concepts of differential invariant and joint invariant, which we shall define in precise terms shortly. In many cases, the existence of invariants is guaranteed by Frobenius' theorem which equates the number of functionally independent invariants to the difference between the dimension of the space acted on and the orbit dimension.

2.3 Noether's Theorem

Noether's Theorem states that if the Lagrangian function for a physical system is not affected by changes in the coordinate system used to describe them, then there will be a corresponding conservation law. For example, if the Lagrangian is independent of the location of the origin then the system will preserve (or conserve) linear momentum.

Application of Noether's theorem allows physicists to gain powerful insights into any general theory in physics, by just analyzing the various transformations that would make the form of the laws involved invariant. For example:

1-the invariance of physical systems with respect to spatial translation (in other words, that the laws of physics do not vary with locations in space) gives the law of conservation of linear momentum;

2-invariance with respect to rotation gives the law of conservation of angular momentum;

3-invariance with respect to time translation gives the well-known law of conservation of energy

In quantum field theory, the analog to Noether's theorem, the Ward–Takahashi identity, yields further conservation laws, such as the conservation of electric charge from the invariance with respect to a change in the phase factor of the complex field of the charged particle and the associated gauge of the electric potential and vector potential

The Noether charge is also used in calculating the entropy of stationary black holes.

We now state Noether theorem:

Suppose the coordinates $\{q_i\}$ are a function of a continuous parameter s. if $\frac{dl}{ds}$ =o then $\frac{dH}{dt} = 0$ where H= $\sum H_i$ and $H_i = p_i(\partial q_i/\partial s)$

Proof:

Consider a quantity $(\partial q_i/\partial s)$ and its product with the corresponding momentum p_i. Call this product H_i; i.e. $H_i = p_i(\partial q_i/\partial s)$

Now consider the time derivative of H_i:

$$\frac{dH_i}{dt} = (\frac{dp_i}{dt})(\frac{\partial q_i}{\partial s}) + p_i \frac{d(\frac{\partial q_i}{\partial s})}{dt}$$

The order of the differentiations (by t and by s) in the second term on the right may be interchanged to give

$$\frac{d(\frac{\partial q_i}{\partial s})}{dt} = \frac{\partial(\frac{dq_i}{dt})}{\partial s} = \frac{\partial v_i}{\partial s}$$

Since $\frac{dp_i}{dt} = \frac{\partial L}{\partial v_i}$ and $p_i = \frac{\partial L}{\partial q_i}$ the time derivative of H_i reduces to

$$\frac{dH_i}{dt} = (\frac{\partial L}{\partial q_i})(\frac{\partial q_i}{\partial s}) + (\frac{\partial L}{\partial v_i})(\frac{\partial v_i}{\partial s}) = \frac{\partial L}{\partial s}$$

But the right-hand-side of this equation is merely the rate of change of L having to do with the effect of a change in s. It is the change in L which occurs as a result of the effect of the change in s on q_i and v_i. A change in s may affect all of the coordinates and not just one. If everything is summed over the n coordinates the result is the total derivative of L with respect to s; i.e., The left-hand-side is just $\frac{dH}{dt}$ where $H = \sum H_i$

Thus if L is independent of s; i.e., $\frac{dL}{ds} = 0$, then $\frac{dH}{dt} = 0.$ and thus H is constant over time; i.e., H is conserved.

3. symmetries and invariants :

3.1 Symmetry

In attacking mathematical problems, one should always look for symmetries that can be exploited to simplify matters. A typical exploitation of symmetry, which occurs in studying a boundary value problem, is to observe that the problem considered has rotational invariance and therefore use polar coordinates. Other kinds of symmetry are more subtle, such as the *duality* (or auto orphism) inherent in projective plane geometry, interchanging the roles of points and lines and thus yielding for free a new theorem "dual" to any given one. In the case of differential equations, the discovery of a symmetry may enable one to reduce the number of variables e.g. transform from a partial to an ordinary differential equation, or from an ordinary differential equation to one of lower order.(These observations were the point of departure for the researches of Sophus Lie.) . In physics A symmetry of a physical system is a physical or mathematical feature of the system (observed or intrinsic) that is "preserved" under some change.

In real-world observations. For example, temperature may be constant throughout a room. Since the temperature is independent of position within the room, the temperature is *invariant* under a shift in the measurer's position. Similarly, a uniform sphere rotated about its center will appear exactly as it did before the rotation. The sphere is said to exhibit spherical symmetry. A rotation about any axis of the sphere will preserve how the sphere looks.

3.2 Differential invariant

A differential invariant is an invariant for the action of a Lie group on a space that involves the derivatives of graphs of functions in the space. Differential invariants are fundamental in projective differential geometry, and the curvature is often studied from this point of view. Differential invariants were introduced in special cases by Sophus Lie in the early 1880s and studied by Georges Henri Halphen at the same time. Lie (1884) was the first general work on differential invariants, and established the relationship between differential invariants, invariant differential equations, and invariant differential operators. The simplest case is for differential invariants for one independent variable *x* and one dependent variable *y*. Let *G* be a Lie group acting on R^2. Then *G* also acts, locally, on the space of all graphs of the form $y = f(x)$. Roughly speaking, a *k*-th order differential invariant is a function

$$I\left(x, y, \frac{dy}{dx}, \ldots, \frac{d^k y}{dx^k}\right)$$

Depending on *y* and its first *k* derivatives with respect to *x*, that is invariant under the action of the group. The group can act on the higher-order derivatives in a nontrivial manner that requires computing the *prolongation* of the group action. The action of *G* on the first derivative, for instance, is such that the chain rule continues to hold: if $(\overline{x}, \overline{y}) = g \cdot (x, y),$ then

$$g \cdot \left(x, y, \frac{dy}{dx}\right) \stackrel{\text{def}}{=} \left(\overline{x}, \overline{y}, \frac{d\overline{y}}{d\overline{x}}\right).$$

Similar considerations apply for the computation of higher prolongations. This method of computing the prolongation is impractical, however, and it is much simpler to work infinitesimally at the level of Lie algebras and the Lie derivative along the *G* action.

More generally, differential invariants can be considered for mappings from any smooth manifold *X* into another smooth manifold *Y* for a Lie group acting on the Cartesian product $X \times Y$. The graph of a mapping $X \to Y$ is a sub manifold of $X \times Y$ that is everywhere transverse to the fibers over *X*. The group *G* acts, locally, on the space of such graphs, and induces an action on the *k*-th prolongation $Y^{(k)}$ consisting of graphs passing through each point modulo the relation of *k*-th order contact. A differential invariant is a function on $Y^{(k)}$ that is invariant under the prolongation of the group action.

3.3 Joint invariants

Joint invariants appear when a transformation group acts simultaneously on several different spaces or, more typically , on a multiple copies of the same space. More specifically ,suppose G is a fixed group which acts on the spaces $X_1, \ldots, X_m$. Then there is a naturally induced action of G on the Cartesian product space $X_1 x, \ldots, \text{x} X_m$ given by $g.(X_1, \ldots, X_m) = (g.X_1, \ldots, g.X_m)$

Definition :

A joint invariant is merely an invariant function

$J : X_1 \times \ldots \times X_m \longrightarrow R$

for a Cartesian product action of a group G.

In other words $J(g.X_1, \ldots, g.X_m) = J(X_1, \ldots, X_m)$ for all $g \in G$, $x_i \in X_I$

4. solving variation via invariants

Invariance has many aspects, we can look at them from different fields all of them doing the same thing (simplifying matters), and here are some examples

4.1Examples

Example1: Let E be a measurable set of real numbers of positive measure. Prove that the set D := {x - y: x and y in E} contains an interval.

Solution: We'll transform the given information about *sets* to corresponding information about *functions*, by associating to E its "characteristic function" f, defined by

f(x) = 1 if x is in E, 0 otherwise.

Now, for fixed t, the function

g(x) = f(x+t)f(x)

equals 1 if both x and x+t are in E, and 0 otherwise. Thus, if this function g(x) is not identically 0, there is an x for which x and x+t are in E, and hence t is in D. From here on assume that E is bounded. Certainly g(x) is not identically 0 if its integral w.r.t. x is positive. But h(t):= Int[f(x+t)f(x) dx] is a continuous function of t and, since h(0) equals the measure of E, which is positive, g is positive for all t with |t| sufficiently small; so those t belong to D,

Example 2: the famous "Liouville theorem": A bounded entire function is constant.

Proof: Suppose f is entire and $|f(z)| \leq M$ for all complex z. We'll assume as known the maximum modulus theorem, and its immediate consequence, *Schwarz' lemma*: If g is holomorphic in the unit disk D and |g| is bounded by 1, and g(0)= 0, then $|g(z)| \leq |z|$. (we grasp at once that the Schwarz lemma is the "high ground" for Liouville's theorem (and many others!).

So, define $g(z) = \frac{(fR_z - f\ (0))}{2M}$, where R is an arbitrary positive number. Since Schwarz' lemma is applicable to g we get $|f\ (R_z) - f\ (0)| \leq 2M\ |z|$ Hence, for any complex number w, setting z = w/R in the last inequality, we obtain $|f(w) - f(0)| \leq 2M\ |w|/R$.

Since this is valid for all positive R, we conclude that f(w) - f(0) vanishes identically**.**

Example3: Solve the heat equation $u_t = u_{xx}$ subject to the initial conditions that u(x,0) = H(x), H denoting the "Heaviside function": H(x) = 0 for x < 0 and 1 for x > 0.

Solution: If u is a solution, define $v(x,t) = u(sx, s^2 t)$ where s denotes a positive parameter. Then, $v_t = v_{xx}$ and v(x,0) = u(sx,0) = H(sx) = H(x), so v satisfies the heat equation with the same initial conditions as u. If we take for granted that the problem has a unique solution then we can conclude v = u, that is $u(x,t) = u(sx, s^2 t)$ identically w.r.t. the parameter s. We have discovered a symmetry of u! Since this relation holds identically in t, x, and s we may set $s = t^{-1/2}$ and obtain

$$u(x, t) = u(x\ t^{-1/2}, 1)$$

showing that u has the form $f(x\ t^{-1/2})$ for some function f of one variable. Thus, exploitation of the symmetry in the initial value problem has narrowed our search to one for the *univariate* function f. The condition that $f(x\ t^{-1/2})$ satisfies the heat equation gives that f(y) satisfies the ordinary differential equation f''(y) + (y/2)f'(y) = 0.

Example3 : Suppose that we have the PDE $\Delta(x, D^{\propto}u) = 0$, in n variables. Suppose that there exists a symmetry group *G* of the PDE and that the orbits of *G* form a submaifold of dimension *p* < *n* : Then the PDE $\Delta(x, D^{\propto}u) = 0$ can always be reduced under a change of variables to a PDE In *n* − *p* variables. In the literature it is common to write the reduced equation as $\Delta / G\ (y, D^{\propto}v) = 0$, where y and *v* are the new variables given y the change of variables. The key to the method is finding invariants of the group action.

Suppose that we have a one parameter group which is generated by a vector field. To determine the action of *v* on a function *f* , we can form the Lie series

$$exp\ (\epsilon v)\ f(x) = f(x) + \epsilon v(f(x)) + ½\epsilon^2 v^2 (f(x)) + \ldots$$

If *f* is invariant under the action of *v* then $f(x) = exp\ (\epsilon v)\ f(x)$ which implies that *v* (*f*) = 0 . This means that the invariants are found y solving the first order PDE

$$\sum_{i=1}^{n} \varepsilon_{i\ (x,u)} \frac{\partial f}{\partial x_i} + \Phi(x,u) \frac{\partial f}{\delta u} = 0$$

i

This can be done by the method of characteristics. We illustrate the procedure finding invariants for some vector fields.

Example4 : Let

$$v = x\ \frac{\partial}{\partial y} - y\ \frac{\partial}{\partial x}.$$

By the method of characteristics we see that the general solution of this PDE is $f = G(x^2 + y^2)$, for *G* an arbitrary differential equation.

Example 5: The one dimensional heat equation $u_t = u_{xx}$ has the symmetry

$$v = 4xt\ \frac{\partial}{\partial x} + 4t^2 \frac{\partial}{\partial t} - (x^2 + 2t)u\ \frac{\partial}{\partial u}.$$

We find invariants of the group generated by *v* by solving

$$\frac{dx}{4xt} = \frac{dt}{4t^2} = -\frac{du}{(x^2 + 2t)u}$$

To demonstrate the method ,we will solve this in steps. First, solving

$$\frac{dx}{4xt} = \frac{dt}{4t^2}$$

Gives *ln x* = *ln t* + *c*. Therefore *c* = *ln* (*x* / *t*).

We could take η = *ln* (*x* / *t*), but since we may actually take any function of *ln* (*x* / *t*) for η, it makes sense just to write $\eta = e^{\ln(x/t)} = x/t$.

Notice that we then have $x = \eta t$. Returning to the previous equation we have to solve

$$\frac{dt}{4t^2} = -\frac{du}{(\eta^2 t^2 + 2t)u}$$

Integration leads to

$$\ln u = -\frac{1}{4}\eta^2 t - \frac{1}{2}\ln t + D.$$

Where *D* is the result of combining the constants of integration from both sides of the equation. Since $\eta = x/t$ we have

$$D = \ln(\sqrt{t}u) + \frac{x^2}{4t}.$$

This gives us our second invariant . In fact, any function of *D* will be a second invariant.

Let us take $v = e^D$ as our second invariant. That is, we set

$$v = \sqrt{t}e^{-x^2/4t}u$$

To be the second invariant. Notice that the two sets of invariants we have obtained are functionally independent. Let us find the group invariant solutions for the heat equation $u_t = u_{xx}$ where the group action is generated by the vector field $v = 4xt\frac{\partial}{\partial x} + 4t^2\frac{\partial}{\partial t} - (x^2 + 2t)u\frac{\partial}{\partial u}$.

We found two invariants, namely $\eta = x/t$ and $v = \sqrt{t}e^{-x^2/4t}u$, let our change of variables be $v = x/t$, and $v = \sqrt{t}e^{-x^2/4t}u$.

Applying the chain rule we have

$$u_t = \left(\frac{(-2t + x^2)v(y) - 4x v'(y)}{4t^{\frac{5}{2}}}\right)e^{-x^2/4t}$$

Turning to the x derivatives gives

$$u_{xx} = \left(\frac{(-2t + x^2)v(y) - 4x v'(y) + 4x v''(y)}{4t^{\frac{5}{2}}}\right)e^{-x^2/4t}$$

So using our expression for u_t and u_{xx}, the heat equation becomes $v''(y) = 0$. The general solution is just v(y) = *Ay* +*B* . Hence the group invariant solutions are of the form

$$u(x,t) = \frac{1}{\sqrt{t}}e^{-x^2/4t}\left(A\frac{x}{t} + B\right)$$

Taking *A*=o, $B = \frac{1}{\sqrt{4\pi}}$ will give the fundamental solution of the heat equation.

We use the invariants to rewrite the PDE. We saw that $\eta = x^2 + y^2$ is an invariant of the rotation group *so*(2) in the plane. The Laplace equation Δu =0 has *so*(2) as a group of symmetries.

Let $r = x^2 + y^2$ be our invariant .We look for a solution of the Laplace equation of the form $u(x, y) = U(x^2 + y^2) = U(r)$. Then by the chain rule Laplace's equation in the plane therefore becomes

$$\Delta u = 4\frac{du}{dr} + 4r\frac{d^2U}{dr^2} = 0.$$

This ODE is called the reduced equation , because we have reduced a PDE in two variables to an ODE.
We may solve this to obtain $U(r) = A\ln r + D,$ where D is a constant of integration. This is the family of solutions of the Laplace equation invariant under rotations . If we take D =0 and A= 1/4 π we obtain the solution

$$U(r) = \frac{1}{4\pi}\ln r = \frac{1}{4\pi}\ln(x^2 + y^2).$$

Which is the fundamental solution of the two dimensional Laplace equation.
What would happen if we chose a different invariant for the change of variables? We could equally have picked $r = \sqrt{x^2 + y^2}$ for the Laplace equation. Doing this , we would obviously arrive at a different ODE. However the ODE which we arrived at would be equivalent to the previous one under the simple change of variables $s \rightarrow \sqrt{r}.$ This is a special case of the more general solution . If η is an invariant and $\xi = f(\eta)$ is another invariant, then the reduced equations we obtained by using η and ξ respectively as change of variables , will always be equivalent under the change of variables $\eta \rightarrow f(\eta)$.

5. conclusion

So we have seen that simplifying matters is an effective tool in solving problems in mathematics and in life. One of the key ideas mathematical problem solving is exploiting invariance. They always simplify matters. So we utilized invariance in solving variations. And we have seen that invariance has many aspects, all of them doing the same thing. The examples given illustrates the idea.

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