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RESEARCH ARTICLE

CUBIC CONVERGENT MODIFIED NEWTON'S METHOD.

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Abstract

In this paper, we suggest an iterative method which is a modified version of Newton's method and it is shown that this method has a cubic rate of convergence.

Introduction:-

We seek the real solution of the equation

$$f(x)=0 \tag{1.1}$$

Where $f(x)$ may be algebraic, transcendental or combination of both. All the iterative methods involve transforming the given equation $f(x)=0$ into the form $x=\phi(x)$ and generating a sequence of approximations defined by

$$x_{n+1}=\phi(x_n) \tag{1.2}$$

(n=0,1,2,….)

starting with x_0. It is well known that this sequence converges, if

$$|\phi'(x)|<1 \text{ for all x in I} \tag{1.3}$$

Where I be an interval containing the true solution and x_0 is chosen in I.

A variant of Newton's method with accelerated third order convergence suggested by S. Weerakoon and T. G. I. Fernando[4] defined by

$$x_{n+1}=x_n-\frac{2f(x_n)}{f'(x_n)+f'(x_n^*)} \tag{1.4}$$

(n=0, 1, 2…)

Where $x_n^*=x_n-\dfrac{f(x_n)}{f'(x_n)}$

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which has a cubic convergence.

The method (1.4) approximates the indefinite integral of the derivative of the function involved in the Newton's method i.e.,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \tag{1.5}$$

(n=0, 1, 2…)

by trapezoid instead of a rectangle thus reducing the error in the approximation.

In section 2, we discuss the Modified Newton's method and where as in section 3, the rate of convergence of this method is obtained. Few numerical examples are considered in the concluding section.

Modified Newton's Method:-

Let x_0 be the initial approximation which is in the vicinity of the real root $'\alpha'$ of the eqn. (1.1) and

$$x_0^* = x_0 + h \tag{2.1}$$

be the next approximation. Then, by finding the point of intersectionof the tangent with the x-axis at the point (x_0, y_0) as in the case of the Newton's method, one can have

$$x_0^* = x_0 - \frac{f(x_0)}{f'(x_0)} \tag{2.2}$$

If assuming 'h' of (2.1) is small enough and higher powers of h are negligible, then $f(x_0^*)$ will almost be negligible. Now, we define

$$x_1 = x_0^* - \frac{f(x_0^*)}{f'(x_0)} \tag{2.3}$$

In similar manner, the second approximate can be obtained as

$$x_2 = x_1^* - \frac{f(x_1^*)}{f'(x_1)} \tag{2.4}$$

Where $x_1^* = x_1 - \frac{f(x_1)}{f'(x_1)}$

In general, the Modified Newton's method can be defined as

$$x_{n+1} = x_n^* - \frac{f(x_n^*)}{f'(x_n)} \tag{2.5}$$

(n=0, 1, 2…)

Where $x_n^* = x_n - \frac{f(x_n)}{f'(x_n)}$

Algorithm 2.1: For a given x_0, compute the approximate solution x_{n+1} by iterative scheme.

$$x_{n+1} = x_n^* - \frac{f(x_n^*)}{f'(x_n)} \tag{2.6}$$

(n=0, 1, 2…)

Where x_n^* is as given in (2.5)

This algorithm is free from second derivative and requires two functional evaluations and one of its first derivatives.

Convergence Analysis:-

Theorem 3.1:

Let $\alpha \in D$ be a single zero of sufficiently differentiable function $f : D \subset R \to R$ for an open interval D. If x_0 is in the vicinity of α, then algorithm 2.1 has third order convergence.

Proof: If $'\alpha'$ be the exact solution of the eqn. (1.1), then

$$f(\alpha) = 0 \quad (3.1)$$

Let e_{n+1} and e_n be the errors at $(n+1)^{th}$ and n^{th} stages and let x_{n+1} and x_n be the $(n+1)^{th}$ and n^{th} approximations to the root $'\alpha'$ of the eqn. (1.1). Therefore, we have

$$x_{n+1} = e_{n+1} + \alpha \quad (3.2)$$

$$x_n = e_n + \alpha \quad (3.3)$$

Now,

$$f(x_n) = f(\alpha + e_n) = f(\alpha) + f'(\alpha)e_n + \frac{f''(\alpha)}{2!}e_n^2 + \frac{f'''(\alpha)}{3!}e_n^3 + O(e_n^4)$$

$$= f'(\alpha)\left[e_n + \frac{1}{2!}\frac{f''(\alpha)}{f'(\alpha)}e_n^2 + \frac{1}{3!}\frac{f'''(\alpha)}{f'(\alpha)}e_n^3 + O(e_n^4)\right]$$

$$= f'(\alpha)\left[e_n + c_2e_n^2 + c_3e_n^3 + O(e_n^4)\right] \quad (3.4)$$

Where $c_j = \frac{1}{j!}\frac{f^j(\alpha)}{f'(\alpha)}$

(j=2, 3, 4…)

$$f'(x_n) = f'(\alpha + e_n) = f'(\alpha) + f''(\alpha)e_n + \frac{f'''(\alpha)}{2!}e_n^2 + O(e_n^3)$$

$$= f'(\alpha)\left[1 + \frac{f''(\alpha)}{f'(\alpha)}e_n + \frac{1}{2!}\frac{f'''(\alpha)}{f'(\alpha)}e_n^2 + O(e_n^3)\right]$$

$$= f'(\alpha)\left[1 + 2c_2e_n + 3c_3e_n^2 + O(e_n^3)\right] \quad (3.5)$$

Now again,

$$x_n^* = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$= \alpha + e_n - \frac{\left[e_n + c_2e_n^2 + c_3e_n^3 + O(e_n^4)\right]}{\left[1 + 2c_2e_n + 3c_3e_n^2 + O(e_n^3)\right]}$$

$$= \alpha + e_n - \left[e_n + c_2e_n^2 + c_3e_n^3 + O(e_n^4)\right] \times \left[1 + 2c_2e_n + 3c_3e_n^2 + O(e_n^3)\right]^{-1}$$

$$= \alpha + e_n - \left[e_n - c_2e_n^2 + (2c_2^2 - 2c_3)e_n^3 + O(e_n^4)\right]$$

$$= \alpha + c_2e_n^2 + (2c_3 - 2c_2^2)e_n^3 + O(e_n^4) \quad (3.6)$$

and,

$$f(x_n^*) = f[\alpha + c_2 e_n^2 + (2c_3 - 2c_2^2)e_n^3 + O(e_n^4)]$$
$$= f(\alpha) + f'(\alpha)\left[c_2 e_n^2 + (2c_3 - 2c_2^2)e_n^3 + O(e_n^4)\right]$$
$$= f'(\alpha)\left[c_2 e_n^2 + (2c_3 - 2c_2^2)e_n^3 + O(e_n^4)\right] \tag{3.7}$$

Adding (3.4) and (3.7), we get

$$f(x_n) + f(x_n^*) = f'(\alpha)\left[e_n + 2c_2 e_n^2 + (3c_3 - 2c_2^2)e_n^3 + O(e_n^4)\right] \tag{3.8}$$

Dividing (3.8) by (3.5), we get

$$\frac{f(x_n) + f(x_n^*)}{f'(x_n)} = \frac{\left[e_n + 2c_2 e_n^2 + (3c_3 - 2c_2^2)e_n^3 + O(e_n^4)\right]}{\left[1 + 2c_2 e_n + 3c_3 e_n^2 + O(e_n^3)\right]}$$
$$= \left[e_n + 2c_2 e_n^2 + (3c_3 - 2c_2^2)e_n^3 + O(e_n^4)\right]$$
$$\times \left[1 + 2c_2 e_n + 3c_3 e_n^2 + O(e_n^3)\right]^{-1}$$
$$= \left[e_n + 2c_2 e_n^2 + (3c_3 - 2c_2^2)e_n^3 + O(e_n^4)\right]$$
$$\times \left[1 - 2c_2 e_n + (4c_2^2 - 3c_3)e_n^2 + O(e_n^3)\right]$$
$$= e_n + (2c_2 - 2c_2)e_n^2 + (4c_2^2 - 3c_3 - 4c_2^2$$
$$+ 3c_3 - 2c_2^2)e_n^3 + O(e_n^4)$$
$$= e_n - 2c_2^2 e_n^3 + O(e_n^4) \tag{3.9}$$

From (2.5)

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{f(x_n^*)}{f'(x_n)}$$
$$= x_n - \frac{f(x_n) + f(x_n^*)}{f'(x_n)}$$
$$x_{n+1} = x_n^* - \frac{f(x_n^*)}{f'(x_n)}$$

$\therefore$ from (3.2), (3.3) and (3.9), we have

$$\alpha + e_{n+1} = \alpha + e_n - e_n + 2c_2^2 e_n^3 + O(e_n^4)$$
$$\Rightarrow e_{n+1} \propto O(e_n^3)$$

Hence, the method (2.5) has a third order convergence.

Numerical Examples:-

We consider few numerical examples considered by S. Weerakoon and T. G. I. Fernando [4] and by B.S. Grewal [5] and the method (2.5) is compared with the methods (1.4) and (1.5). The computational results are tabulated below and the results are correct up to an error less than 0.5×10^{-7}.

Table 4.1:-

Function	x_0	i			NOFE			Root
$f(x)$		NM (1.5)	VNM (1.4)	MN (2.5)	NM (1.5)	VNM (1.4)	MN (2.5)	
(1) x^3+4x^2-10	2.5	6	4	4	12	12	12	1.36523001
(2) $\sin^2(x)-x^2+1$	3 3.5 9	6 6 8	3 4 5	3 4 5	12 12 16	9 12 15	9 12 15	1.404492
(3) x^2-e^x-3x+2	-3.5 2.6	6 6	4 4	4 4	12 12	12 12	12 12	0.25753028
(4) $\cos(x)-x$	3 3.5	6 13	12 9	4 5	12 26	36 27	12 15	0.7390851
(5) $(x-1)^3-1$	-1.9 2.6 6.3	9 6 9	29 4 6	6 4 6	18 12 18	58 12 18	18 12 18	2
(6) x^3-10	3.5	6	4	4	12	12	12	2.1544346
(7) $2x-\log_{10}x-7$	3.9	3	3	2	6	6	4	3.789278
(8) $xe^x-\cos x$	1.2	6	4	4	12	12	12	0.5177564
(9) $2x-\log x-6$	3.6 37	3 5	2 4	2 3	6 10	6 12	6 9	3.646945
(10) $4e^{-x}\sin x-1$	2	5	4	3	10	12	9	1.364958

NM- Newton's Method
VNM – Variant of Newton's Method
NOFE – Number of Function Evaluations MN – Modified Newton's Method
i-Number of iterations to approximate the root to 7 decimal places

Conclusion:-

It is evident from the above computational results that the method (2.5) has a third order convergence and requires lesser or the same number of total functional evaluations compared to the method (1.5) & (1.4) and doesn't need tocompute $f'(x_n^*)$ at each step as in the case of the method (1.4).

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