Problem Solving and Hints Posing

Abstract

Twenty teachers were asked what they think about problem solving, how they manage problem solving in their class, and how they pose hints to problems as scaffolds to the students. This paper reports their answers and analyzes them. According to the teachers, problem solving makes the teaching more interesting, is fun for the students and increases motivation to study mathematics. Problem solving develops mathematical thinking and creativity, including "thinking outside the box". Problem solving allows deep understanding of the material, develops the ability to cope with complicated problems and can be used to identify the potential of the students. On the other hand, the teachers mentioned lack of time and increased teaching and learning load as disadvantages of problem solving. Some teachers felt that problem solving is useful only for teaching excellent students in small groups

The teachers were asked to suggest interesting problems and hints as scaffolds to solve the problems. The problems and the hints are described in the paper. The teachers' answers show that the art of posing problems and hints should be developed. This should be done in all problem-solving courses for pre-service teachers and in professional development programs for teachers.

Key words: problem solving, problem posing, hint posing, scaffolding

1. Introduction

This paper describes and analyzes a questionnaire given to 20 middle school teachers. The research questions were:

1. What is the teachers' attitude to problem solving?

2. (How) do they integrate problem solving in their teaching?

3. What are the students' difficulties in problem solving?

4. How do the teachers pose hints to help the students to cope with these difficulties?

The structure of the paper is as follows: The theoretical background is based on literature on problem solving, problem posing and hints as scaffolds. The methodology section consists of the questionnaire given to the teachers. An important part of the questionnaire was that the teachers were asked to suggest a problem and a related hint. The data section consists of the answers of the teachers, and of the problems and the hints that they suggested. The paper is concluded with a discussion of the teachers' answers. For two of the problems one hint may not be sufficient and additional hints are suggested.

2. Theoretical Background

2.1 Problem solving

Problem solving lies at the heart of mathematics and is an important element of mathematics teaching (Guberman and Leikin, 2013; NCTM, 2000). According to Hmelo-Silver, Duncan and Chinn (2007), in contrary to exercise, "a problem is a task that there is no complete algorithm for obtaining a solution". Polya (1957, 1962), proposed the following stages in solving a problem: understanding the problem; solution plan; execution of the plan; retrospective review – evaluation and review. When students learn how to solve a problem, it may become routine. This suggests the importance of posing new non-routine problems. Disclaimer: solving exercises is an essential and crucial part of teaching mathematics. Saying that, problem solving has many advantages. According to Halmos (1995), it is fun to solve problems, and good problems are a good way to learn. According to Leikin and Guberman (2023), problem solving develops mathematical thinking and creativity and improves mathematical skills and motivation. On the other hand, some researchers point out that problem solving is time consuming and may frustrate and frighten some students (Di Leo et al., 2019). This may suggest that problem solving is suitable only for gifted and advanced students. Many researchers, including the authors of this paper, do not agree. Koichu (2004) showed that it is worthwhile and possible to integrate challenging problems into teaching all

(not only gifted) students. In general, some students have difficulty solving non-routine problems, because they do not have the ability to apply the existing knowledge and use it in the new situation (Star and Rittle-Johnson, 2008).

Mahagna, Berman and Leikin (2023) found that involving ordinary students in problem-solving positively affected the students' curricular achievements in mathematics and their ability to solve unfamiliar enrichment problems.

2.2 Problem posing

Problem posing is important in teaching and learning mathematics (NCTM, 2000). According to Halmos (1995) a good problem should not be hard and not easy.

Klinstern, Koichu and Berman (2013) report on a study conducted with many high-school teachers. They invited 500 teachers to respond to a survey asking what sources they use to choose problems and what are their reasons for posing problems, when they pose their own problems. Thirty percent of the teachers (N=151) responded. It is not surprising that 109 teachers marked textbooks as the main source. It is less obvious that out of 146 teachers who referred to "posing my own problems", 43 wrote that they do it "sometimes", 31 – "often" and 10 - "almost always" (29 teachers wrote that they "almost never" pose problems, and 33 do it "rarely").

The reasons that the teachers gave for problem posing were: to adapt textbooks problems to the needs and ability of the students; for evaluation; to connect several topics; to connect the learned material to real life; to explain new topics.

Posed problems can be special cases of research problems. The converse is also true. Problems used for teaching can be generalized to research problems, e.g. Halbeisen and Hungerbühler (1995).

2.3 Hints as scaffolds

Wood, Bruner and Ross (1976) described scaffolding as a "process that enables a child or novice to solve a problem, carry out a task or achieve a goal which would be beyond his unassisted efforts" (p. 90).

Rachamim, Berman and Koichu (2022) describe a case study of four teachers that were asked to design new geometry tasks for their students. They were supported by scaffolds such as modeling, reflection, articulation and worked examples.

Hints are an important scaffold in problem solving. The paper describes hints given by the teachers that participated in the study.

3. Methodology

3.1 Research population

The research population consisted of 20 middle school teachers. Every teacher is denoted, in a random way, by a number.

Information about their education and experience in teaching is given in the following table.

Table 1: The teachers' education and seniority

Teacher	Education	Number of years of teaching mathematics
1	B.A in Law, M.A in Math Education	20
2	B. Sc in Mathematics, M.A in Math Education	16
3	B.A in Mathematics, M.A in Math Education	29
4	B.Ed in Math Education, M.Ed in Math Education	24
5	B.A in Economics, M.Ed in Math Education	8
6	B.Ed in Math Education, M.A in Math Education	17
7	B.A in Social Science	4

8	B.A in Mathematics, M.A in Math Education	25
9	B.Ed in Math Education	16
10	B.Ed in Math Education, M.Ed in Science and Technology Education.	12
11	B.Ed in Math Education	13
12	B.Ed in Math Education	3
13	B.A in Mathematics, M.A in Math Education	38
14	B.Ed in Math Education	17
15	B.A in Mathematics and Computer Science	32
16	B.A in Statistics, M.A in Math Education	17
17	B.A in Mathematics, M.A in Math Education	25
18	B.A in Mathematics and Computer Science, M.A in Law	16
19	B.Ed in Math Education, M.A in Math Education	18
20	B.A in Biology, M.A in Medical Sciences	14

3.2 The research tool

The research tool was a questionnaire in which the teachers were asked to describe their attitude toward problem solving and their classroom-solving management. They were also asked to give an example of an interesting problem and suggest a hint to help the students to solve it. Here is the questionnaire:

Please answer the following questions

A. Personal information

1. In what disciplines did you obtain your academic degrees?

2. How many years do you teach mathematics?

B. Teachers' attitude toward problem solving

1. What, in your opinion, are the advantages and disadvantages of using problems in teaching mathematics?

2. What, in your opinion, cause the difficulties that students encounter in solving problems?

C. Classroom problem solving management

1. Please mark the appropriate statement

   I use problems in my teaching:

   a. quite a lot.

   b. every week.

   c. sometimes.

   d. seldom (I seldom use problems).

   e. only for advanced students.

2. Please mark the appropriate statement

   The problems that I use in my teaching are:

   a. only curriculum related.

   b. only for enrichment.

   c. mostly curriculum related.

   d. mostly for enrichment.

   e. both curricula related and for enrichment.

3. Please mark the appropriate statements:

   a. The students solve the problems alone.

   b. The students solve the problems in pairs.

   c. The students solve the problems in a larger group.

4. Please mark the appropriate statements:

   a. I give hints with the problems.

b. I give the students time to solve the problems, and then, if necessary, help them with hints.

c. I do not give hints.

D. Problems and hints

Please give an example of an interesting problem, and possible hints.

Thank you very much for participating in the research. Would you like to get a copy of the research findings?

4. Data

4.1 Teachers' attitudes

The teachers mentioned several advantages and disadvantages of problem solving and difficulties in solving problems.

Disadvantages

a1. Takes time needed to cover the curriculum

a2. Some students are not active during problem solving

a3. Causes frustration

a4. Students who cannot solve the problems may hate mathematics

a5. Increase learning load

a6. Problems frighten some students

a7. Useful only for small groups

a8. Increase the teaching load

a9. Appropriate only for excellent students

Advantages

b1. Develops the ability to think out of the box

b2. Interesting

b3. Develops creativity

b4. Shows the relevance of mathematics

b5. Motivates the learning of mathematics

b6. Is a good way to teach mathematical ideas

b7. Develops the ability to cope with complicated problems

b8. Develops mathematical thinking

b9. Let's the students think in several ways

b10. Breaks the routine learning

b11. Can be used to find the potential of the students

b12. Makes the teaching more diverse

b13. Strengthens the student's self confidence

b14. Integrates knowledge from different areas

b15. Prepares the students for the matriculation exam

b16. Is an excellent way to teach excellent students

b17. Challenges the students

Difficulties

c1. Some students are not used to solve problems that are not related to the learned material

c2. Some students have low confidence in their ability to solve problems

c3. The students are used to solving only problems with known algorithm

c4. For some students the problems are above their level

c5. Difficulty in reading comprehension

c6. Difficulty in understanding and using the data

c7. There are not enough problems in the textbooks

c8. Mathematics taught in school develops technique and not understanding

c9. Challenging problems are useful for good students, but maybe difficult for weak students

c10. The students do not have the skills needed to solve the problems

The answers of the teachers, concerning disadvantages, advantages and difficulties, are listed in Table 2.

Table 2: Teachers' attitudes

Teacher	Disadvantages	Advantages	Causes of difficulties
1	a1	b1, b2, b3, b4, b5	c1, c5
2	a1, a2	b6	c2
3	a1	b2	
4		b7	c3, c5
5	a3	b8	c3
6		b1, b8	c2, c4
7		b7	c5
8	a1	b1, b9	c5
9	a4, a5	b1, b9, b10	c5
10	a1	b11	c5, c6
11	a1, a6	b8, b11	c7
12		b10, b11, b4, b10	c1
13	a1	b8	c1
14		b8, b7, b5	c8
15	a7	b5	c9
16	a1	b5, b14	c1, c10
17	a1	b5, b15	c10
18	a8	b2, b8, b5, b4	c1
19	a3, a9	b16	c5
20		b8, b17	c1

4.2 Classroom problem solving management

The teachers were asked to report the frequency of their using problem solving, whether the problems that they use are related to the subject that they teach, do the students work in groups and how do they use hints.

Frequency of using problem solving

A1. Every week

A2. Sometimes

A3. Seldom

A4. Quite a lot

A5. Once a week

A6. Every lesson

A7. Seldom in ordinary class and quite a lot in excellent class

Curriculum related or enrichment

B1. Mostly curriculum related

B2. Only enrichment

B3. Both curricula related and for enrichment

Working in groups

C1. In simple problems I ask them to work individually and in more complicated problems I ask them to work in groups.

C2. Alone

C3. Groups

C4. Pairs

C4. Sometimes individually and sometimes in groups

C5. Sometimes alone and sometimes in pairs

C6. Sometimes in pairs and sometimes in bigger group

Use of hints

D1. Only when needed

D2. With the problem

D3. If the students fail to solve a problem in class, the problem is given as homework. The solutions are then discussed in the next meeting in class, and then, if needed, hints are given.

The answers of the teachers, concerning problem solving management, are listed in Table 3.

Table 3: Problem solving management

Teacher	Frequency of using problem solving	Curriculum related or enrichment	working groups	Use of hints
1	A1	B1	C1	D1
2	A2	B1	C2	D2
3	A3	B2	C3	D1
4	A4	B3	C4	D2
5	A3	B1	C2	D1
6	A4	B1	C4	D1
7	A3	B3	C3	D2
8	A4	B3	C5	D1
9	A4	B2	C3	D1
10	A5	B3	C2	D1
11	A3	B2	C4	D1
12	A6	B3	C5	D3
13	A4	B3	C4	D1
14	A4	B3	C6	D1

15	A3	B1	C2	D1
16	A3	B3	C4	D1
17	A4	B1	C2	D1
18	A4	B3	C4	D1
19	A7	B3	C4	D1
20	A5	B3	C3	D1

4.3 Problems and hints

The problems and the hints suggested by the teachers are listed in the following table.

Table 4: Problems and hints

Teacher	Problem	Hint
1	The unit digit of a number is 5. Prove that the last two digits of its square are 25.	The number is of the form 10A + 5
2	What is $x + y, if\ 2^x \cdot 2^y = 32$?	$a^m a^n = a^{m+n}$
3	Orit, Gabi and Sofi plan to meet for lunch. They report the height of where they are and decide to meet at the highest place. Orit says that her place is higher than 50 meters, Gabi says that he is 50 meters higher than Orit and Sofi says that the height of her place is twice the height of Orit's. Where will they meet?	Let x be the height of Orit's place. At what heights are Gabi and Sofi?
4	A bird stands on the top of an 8 meters high tree and a turtle stands on the ground, 6 meters from the tree. What is the distance between the bird and	Use the Pythagorean theorem.

the turtle?

5 The functions:

$f(x) = x^2(x-3)(2x+4)$

g(x) = $x^3(x-3)(2x+4)$

h(x) = $x^3(3-x)(2x+4)$

have the same zero points.

Draw the sketches of the functions.

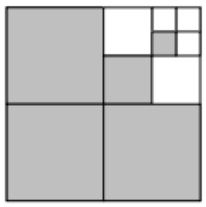
When do the functions vanish?

6 Suggest a business model for a parking lot.

Try different options, for example, fix payment, payment by hours.

7 The following square is painted gray and white.

The area of the square is 1 m^2.



What is the area of the part of the square, painted in gray?

The square is divided into 5 white sub squares and 5 gray sub squares. What is the area of each gray sub square?

8 Prove the roots formula:

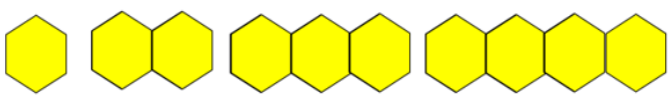
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

for $ax^2 + bx + c = 0$, a≠0.

Compute:

$(x + \frac{b}{2a})^2$

9 The first 4 terms of the sequence of hexagons are:

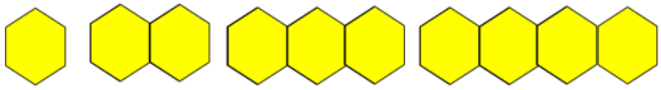


What is the circumference of the general term?

Fill the following table:

Numbers of hexagons	circumference

		1 | 6 2 | 11 3 | 4 |
10	What is $5^x, if\ 1 = 5^{2x+4}$?	What is 2x + 4?
11	You have 12 coins that look the same. One coin is counterfeit and is either heavier or lighter than the other 11. With a balance beam scale, find the counterfeit coin in three weighting*s*.	Solve a similar problem with 3 coins and 2 weightings.
12	Given a balance scale and a bag with 64 kg of nails. How can we weigh 23 kg of nails?	How can we weigh 32 kg?
13	(non)	
14	Find the values of A, B, C, D if $\mathrm{ABCD} \cdot 4 = \mathrm{DCBA}$.	Show that A is an even positive number.
15	(non)	
16	What is the unit digit in $3^{245} + 1$?	What is the unit digit in 3^2?, 3^3?, 3^4?, 3^5?
17	Explain why division by zero is not defined	What is 2x, if x= $\frac{1}{2}$? What is 3x, if x= $\frac{1}{3}$? What is $0 \cdot x$?
18	Divide 1023 eggs into boxes in such a way that for every n<1023, if you are asked to give n eggs you can give some boxes without opening them.	What is the sum 1+2+4+8+…+512?
19	The first 4 terms of the sequence of hexagons are:	How does the

	 What is the circumference of the general term?	circumference increase when a hexagon is added?
20	Given 3 points, A, B, C, what are the points X such the area of XBC is the same area of ABC?	Use the formula for area of a triangle.

5. Summary and discussion

This section is a summary of the data presented above and discuss the hints posed by the teachers.

5.1 The teachers

Fourteen of the teachers have a second degree, 12 of them in math education, one in law (first degree in mathematics and computer science), one in medical sciences (first degree in biology); 4 teachers have first degree in mathematics education, one in mathematics and computer science and one in social sciences. Most of the teachers are experienced. The most experienced has 38 years of seniority, first degree mathematics, and second degree in math education. One teacher who studied math education has 3 years of experience and one teacher who studied social sciences has 4 years of experience.

5.2 The teachers' attitudes

According to the teachers the advantages of problem solving are:

1. It makes the teaching more interesting; it is fun for the students and increases the motivation to study mathematics.
2. It develops mathematical thinking, creativity and "thinking outside the box", it allows deep understanding of the material, it develops the ability to cope with complicated problems.
3. It can be used to identify the potential of the students.

The main disadvantages are:

1. Lack of time.

2. Problem solving increases the teaching load of the teachers and the learning load of the students.
3. Problem solving is appropriate only for excellent students and is useful only when they are taught in small groups.

The main difficulties are:

1. Students are not used to solving problems without a given algorithm.
2. Reading comprehension.
3. Some students have insufficient mathematical background, and some do not believe in their abilities.

5.3 Problem solving management

Frequency of using problem solving

Every lesson - one teacher.

Once a week - three teachers.

Quite a lot - eight teachers.

Quite a lot working with excellent students and seldom with ordinary students - one teacher.

Seldom - seven teachers.

Curriculum related or enrichment

Twelve teachers use both curriculum related problems and enrichment problems; one of them gives an enrichment problem every week.

Six teachers use only problems related to the curriculum and two teachers use problems only as enrichment.

Solving problems in groups

Five teachers want the students to work alone, five teachers allow the students to work alone or in groups, one of them wants the students to work alone on simple problems and in groups on more difficult problems. Four teachers want the students to work in groups; two teachers

want the students to work in pairs, one teacher lets the students work in pairs or in bigger groups and one teacher uses all options.

Use of hints

Most teachers (N=18) give the hint only when needed. One of them gives the hint, if needed, after the problem was given as homework. Two teachers give the hint with the problem.

5.4 Problems and hints

The teachers were asked to suggest interesting problems. Eighteen (out of twenty) did. Most of the problems were chosen from textbooks or from the internet. For example, the 12 coins problem (Problem 11) appeared in many papers (e.g. Guy & Nowakowski, 1995), and even in the New York Times (Antonick, 21.7.2014). Out of the 18 problems, 7 are routine. Nine of the non-routine problems are interesting but not difficult, so they satisfy one of the criteria for being a good problem, (as defined by Halmos, 1995). For these problems, the hints suggested by the teachers are sufficient. Problems 9 and 19 were taken from the same source, but the hints suggested by the teachers are different. Some students understand the problem by writing the data in a table. For them Hint number 9 is helpful. Hint number 19 is sufficient for students who do not need a table or can be used after Hint number 9.

For the weighing problems, Problems 11 and 12, there may be a need for additional hints. After a hint, the students should have enough time to use it and then, if necessary, another hint can be given.

Hints for the 12 coins problem:

Hint 1 was given by the teacher "Solve a similar problem with 3 coins and 2 weighting".

Hint 2: Numerate the coins 1,2, 3, … ,12 and divide them into three equal sets.

Hint 3: Weight the coins {1,2,3,4} against the coins {5,6,7,8} .

In the case that the weights are not equal, there may be a need for another hint.

Hint 4: Weight the coins {1,2,5} against the coins {3,4,6} .

Remark: it is important that the students complete the proof and feel that they solved the problem or participated in solving it.

For the nails problem, the teacher suggested the hint "How can we weigh 32 kg?".

A possible follow up hint: express 23 as some powers of 2 (write the binary form of 23).

It is interesting to point out that the weighting problems were suggested by teachers with B.Ed. degree. We did not find connection between the choice of the problem and the use of the problems in the teaching. The teacher who suggested the 12 coins problem does not give many problems in his teaching. On other hand the teacher who suggested the nails problem gives problems every week.

The lunch problem (Problem 3) is interesting from the problem posing point of view. The information given that Orit's place height is greater than 50 meters guarantees that the problem has a unique solution.

The problem of proving quadratic equation formula (Problem 8) is interesting from the didactical point of view. The teacher asks the students to prove the formula before he does it. This pedagogy is called "problem solving before instruction", see

Suggestions

Challenging problems and hints that help the students to solve the problems are important. Problem posing, and hints should be included in problem solving courses for pre-service teachers and for teachers, in professional development programs.

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