Perfect s-geodetic fuzzy graphs

Abstract

The set of nodes which do not belong to any s-geodetic basis of a fuzzy graph G is the Pseudo s-geodetic set of G and its cardinality is called the Pseudo s-geodetic number of G. In this paper, fuzzy graphs having Pseudo s-geodetic number zero are termed as Perfect s-geodetic fuzzy graphs and some examples of Perfect s-geodetic fuzzy graphs are exhibited. It is proved that complete fuzzy graphs on 2 nodes and fuzzy cycles having each arc of same strength are Perfect s-geodetic fuzzy graphs.

Keywords:s-Geodetic basis, s-Geodetic number, Pseudo s-geodetic set, Pseudo s-geodetic number, Perfect s-geodetic fuzzy graph.

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1 Introduction

Zadeh in 1965 [18] developed a mathematical phenomenon for describing the uncertainties prevailing in day-today life situations by introducing the concept of fuzzy sets. The theory of fuzzy graphs was later on developed by Rosenfeld in the year 1975 [11] along with Yeh and Bang [17]. Rosenfeld also obtained the fuzzy analogue of several graph theoretic concepts like paths, cycles, trees and connectedness along with some of their properties [11] and the concept of fuzzy trees [10],fuzzy interval graphs [7],cycles and co-cycles of fuzzy graphs [8] etc has been established by several authors during the course of time. Fuzzy groups and the notion of a metric in fuzzy graphs was introduced by Bhattacharya [1]. The concept of strong arcs [4] was introduced by Bhutani and Rosenfeld in the year 2003. The definition of fuzzy end nodes and some of their properties were established by the same authors in [2]. The concept of geodesic distance was introduced by Bhutani and Rosenfeld in [3] and using this geodesic distance, Suvarna and Sunitha in [16] brought the concept of

geodesic iteration number and geodesic number of a fuzzy graph into existence and studied some of the properties satisfied by them. The same concepts using μ-distance was introduced by Linda and Sunitha in [5].The concept of sum distance and some of its metric aspects were introduced by Mini Tom and Sunitha in [6]. s-Geodetic iteration number and s-geodetic number of a fuzzy graph based on sum distance was introduced by Sameeha and Sunitha in [13].
The set of nodes of a fuzzy graph $G:(V,\sigma,\mu)$ which do not belong to any s-geodetic basis of G is the pseudo s-geodetic set of G and its cardinality is called the pseudo s-geodetic number of G [14]. In this paper, fuzzy graphs having pseudo s-geodetic number zero are termed as perfect s-geodetic fuzzy graphs and some examples of perfect s-geodetic fuzzy graphs are exhibited along with some of their properties. Complete fuzzy graphs and fuzzy cycles are proved to be perfect s-geodetic fuzzy graphs.

2 Preliminaries

In this section, a brief summary of some basic definitions in fuzzy graphs taken from [3, 4, 9, 10, 16] are given.
A fuzzy graph [9] is a triplet $G:(V,\sigma,\mu)$ where σ is a fuzzy sub set of a set V of nodes and μ is a fuzzy relation on σ. ie, $\mu(u,v) \leq \sigma(u)\wedge\sigma(v)$, $\forall u,v \in V$. We assume that V is finite and non-empty, μ is reflexive (*i.e.*,$\mu(x,x)=\sigma(x), \forall x$) and symmetric (*i.e.*,$\mu(x,y)=\mu(y,x), \forall (x,y)$). Also we denote the under- lying crisp graph by $G^*:(\sigma^*,\mu^*)$ where $\sigma^*=\{u\in V/\sigma(u)>0\}$ and $\mu^*=\{(u,v)\in V\times V/\mu(u,v)>0\}$. Here we assume $\sigma^*=V$. A fuzzy graph is called a complete fuzzy graph [9] if $\mu(u,v)=\sigma(u)\wedge\sigma(v)$ $\forall u,v \in\sigma^*$. A sequence of distinct nodes $u_0,u_1,\ldots,u_n$ such that $\mu(u_{i-1},u_i)>0, i=1,2,\ldots,n$ is called a path P_n [9] of length n. An arc of G with least non-

Zero membership value is the weakest arc of G. The degree of membership of a weakest arc in the path is defined as the strength of the path. The path becomes a cycle if $u_0=u_n$, $n\geq 3$ and a cycle is called a fuzzy cycle [10] if it contains more than one weakest arc. The strength of connectedness [9] between two nodes u and v is the maximum of the strengths of all paths between u and v and is denoted by $CONN_G(u, v)$. The fuzzy graph G is said to be connected if $CONN_G(u,v)>0$ for every u,v in σ^*. An arc (u,v) of a fuzzy graph is called strong [4] if its weight is at least as great as the strength of connectedness of its end nodes u,v when the arc (u,v) is deleted and a $u-v$ path P is called a strong path if P contains only strong arcs.

For any path $P:u_0-u_1-u_2-\ldots-u_n$, **length** of P, $L(P)$, is defined as the sum of the weights of the arcs in P. That is , $L(P)=\Sigma^{n}_{i=1}\mu(u_{i-1},u_i)$.
If $n=0$, define $L(P)=0$ and for $n\geq 1$, $L(P)>0$.
For any two nodes u,v in $G:(V,\sigma,\mu)$, if $P=\{P_i$: P_i is a u-v path, $i=1,2,3,\ldots\}$, then the **sum distance** between u and v is defined as $d_s(u,v) = Min\{L(P_i) : P_i\in P, i = 1,2,3,\ldots\}$[6].

Let S be a set of nodes of a connected fuzzy graph G. The s-geodetic closure[13] (S) of S is the set of all nodes in S together with the nodes that lie on s-geodesics between nodes of S. S is said to be convex if S contains all nodes of every $u-v$ s-geodesic for all u,v in S. i.e, if $(S)=S$.
S is said to be s-geodetic cover (s-geodetic set) of G if $(S)=V(G)$ and any Cover of G with minimum number of nodes is called an s-geodetic basis for G. The s-geodetic number [13] of a fuzzy graph G: (V,σ,μ) is the number of nodes in a s-geodetic basis of G and is denoted by $s-gn(G)$.

The following results have been taken from [16].

Corollary2.1.*[16] For a complete fuzzy graph G on* 2 *nodes,* $s-gn(G)=2$.

Remark2.2.[13] Let C_n, $n\geq 3$, be fuzzy cycles each of whose arcs are having same strength. When n is even, the set of any two s-peripheral nodes is an s-geodetic set of C_n. But when n is odd, no 2 nodes form an s-geodetic set and in fact there exists an s-geodetic set on 3 nodes. Therefore, for cycles having each arc of same strength,

$$\text{s-gn}(C_n)= \begin{cases} 2; & \text{when n is even} \\ 3\ ; & \text{when n is odd} \end{cases}$$

3 Perfect s-geodetic fuzzy graph

In graph theory, the concept of Perfect edge geodetic graph was introduced by Stalin in [15] and in fuzzy graph theory, the concept of Perfect geodesic fuzzy graphs was developed using geodesic distance in [12].
In this paper, the vertex version of this concept in fuzzy graph theory is developed using sum distance and is termed as Perfect s-geodetic fuzzy graph.

Definition3.1.[14] Let G:(V,σ,μ) be a connected fuzzy graph and S be an s-geodetic basis of G. Then the set of nodes which do not belong to any s-geodetic basis of G is the Pseudo s-geodetic set S of G.
The cardinality of Pseudo s-geodetic set S is called Pseudo s-geodetic number and is denoted by $s-gn(G)$.

Definition3.2.A connected fuzzy graph G : (V,σ,μ) is said to be a Perfect s-geodetic fuzzy graph if every node of G lies in any one of the s-geodetic basis of G.
In other words, G is a Perfect s-geodetic fuzzy graph if its pseudo s-geodetic numbers $-\ gn\ (G) = 0$.

Example3.3. Consider the fuzzy graph G given in Fig.1.

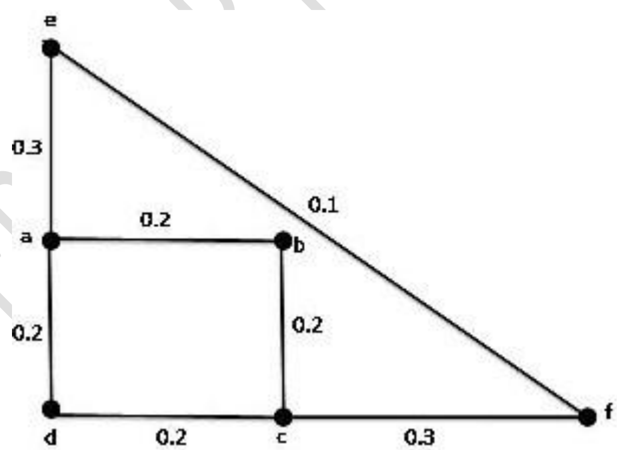

Fig.1

Here, the arc (e,f) is the weakest arc of G. S_1={e,a,c,f} and S_2={e,b,d,f} are both s-geodetic basis for G since (S_1)=(S_2) =V(G).

Then the pseudo s-geodetic set S is $V(G)-\{a,b,c,d,e,f\}=\varphi$ and hence $s-gn(G)$=0. Therefore G is a Perfect s-geodetic fuzzy graph.

Proposition 3.4. If $S_1, S_2, \ldots, S_n$ are the s-geodetic bases of a fuzzy graph $G:(V,\sigma,\mu)$, then the pseudo s-geodetic number $s-gn'(G) = |\cap_{i=1}^{n} S_i^c|$.

Proof. Let S' be the pseudo s-geodetic set of G. To show that $s-gn'(G) = |\cap_{i=1}^{n} S_i^c|$, it is enough to show that $S' = \cap_{i=1}^{n} S_i^c$.

Let v be a node of G such that $v \in S'$. Then by definition 3.1, v does not belong to any s-geodetic basis of G.

i.e, $v \notin S_i \; \forall\, i = 1,2,\ldots,n$.

$\Rightarrow v \in S_i^c \; \forall\, i = 1,2,\ldots,n$.

$\Rightarrow v \in \cap_{i=1}^{n} S_i^c$

$\Rightarrow S' \subseteq \cap_{i=1}^{n} S_i^c$ .... (1)

Conversely, let u be a node of G such that $u \in \cap_{i=1}^{n} S_i^c$.

Then $u \in S_i^c \; \forall\, i = 1,2,\ldots,n$.

$\Rightarrow u \notin S_i \; \forall\, i = 1,2,\ldots,n$.

Hence by definition 3.1, $u \in S'$ and so $\cap_{i=1}^{n} S_i^c \subseteq S'$ .... (2)

From (1) and (2), $S' = \cap_{i=1}^{n} S_i^c$ □

Proposition 3.5. A complete fuzzy graph on 2 nodes is a perfect s-geodetic fuzzy graph.

Proof. By Corollary 2.1, the s-geodetic number of a complete fuzzy graph G on 2 nodes is $s-gn(G) = 2$. Therefore $S = V(G)$ is the unique s-geodetic basis of G and so by Proposition 3.4, the pseudo s-geodetic set $S' = S^c = \varphi$. Thus we get $s-gn'(G) = 0$. Hence by Definition 3.2, G is a perfect s-geodetic fuzzy graph. □

Proposition 3.6. A fuzzy cycle G on n nodes, each of whose arcs are having same strength, is a perfect s-geodetic fuzzy graph.

Proof. Consider the following Cases:

Case (1): n is even.

By Remark 2.2, the s-geodetic number $s-gn(G) = 2$ if n is even. Clearly $S_i = \{v_i, v_{(i+\frac{n}{2}) \bmod n}\}, (1 \le i \le n)$, are the only s-geodetic bases of G. Then by Proposition 3.4, the pseudo s-geodetic set $S' = \varphi$ and so $s-gn'(G) = 0$. Hence G is a perfect s-geodetic fuzzy graph.

Case (2): n is odd.

By Remark 2.2, the s-geodetic number $s-gn(G) = 3$ if n is odd. Clearly

$S_i=\{v_i, v_{(i+\frac{(n-1)}{2})mod\, n}, v_{(i+\frac{(n+1)}{2})mod\, n}\}, (1\leq i\leq n)$ aretheonlys-geodetic basesof$G$.ThenbyProposition3.4,thepseudos-geodeticset$S=\varphi$and so$s-gn(G)$=0.Hence$G$isaperfects-geodeticfuzzygraph. □

4 Conclusion

Thepseudos-geodeticsetofafuzzygraph*G*isdefinedasthesetofnodes of *G* which do not belong to any s-geodetic basis of *G* and its cardinality iscalled the pseudo s-geodetic number of *G*.In this paper, perfect s-geodetic fuzzy graphs are defined to be those fuzzy graphs whose pseudo s-geodetic numberis0.Thatisifallnodesofthefuzzygraph*G*liesinatleastone of the s-geodetic bases of *G*.A complete fuzzy graph on 2 nodes and fuzzy cycles,eachofwhosearcsarehavingsamestrength,arefoundtobeperfect s-geodetic fuzzy graphs.

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REFERENCES

[1] Bhattacharya.P.:Some Remarks on fuzzy graphs, Pattern Recognition Lett., 6 (1987), 297-302.

[2] Bhutani.K.R.,Rosenfeld.A.: Fuzzyendnodesinfuzzygraphs,Inform.Sci. 152(2003), 323-326.

[3] Bhutani.K.R.,Rosenfeld.A.:Geodesics in fuzzy graphs, Electronic Notes in Discrete Mathematics, 15(2003), 51-54.

[4] Bhutani.K.R.,Rosenfeld.A.:Strongarcsinfuzzygraphs,Information Sciences, 152(2003), 319-322.

[5] Linda.J.P.,Sunitha.M.S.:GeodesicandDetourdistancesinGraphsand Fuzzy Graphs, Scholars' Press, (2015).

[6] Mini Tom,Sunitha. M. S.:Sum Distance and Strong Sum Distancein Fuzzy Graphs, LAP Lambert Academic Publishing, 2016, ISBN 10: 3659821969ISBN 13:9783659821967.

[7] Mordeson.J.N.: Fuzzy line graphs, Pattern recognition Lett., 14(1993), 381-384.

[8] Mordeson.J.N.,Nair.P.S.:Cycles and Cocycles of fuzzy graphs, Information Sciences, 90(1996), 39-49.

[9] Moderson. J.N.,Nair.P.S.:Fuzzy Graphs and Fuzzy Hypergraphs, Physica-Verlag, Heidelberg, 2000.

[10] Mordeson.J.N, Yao.Y.Y.: Fuzzy cycles and fuzzy trees, The Journal of Fuzzy Mathematics, 10 (1) (2002), 189-202.

[11] Rosenfeld.A.:Fuzzy graphs, In:L.A.Zadeh, K.S.Fu and M.Shimura(Eds),FuzzySetsandtheirApplications,Academic Press, New York, (1975), 77- 95.

[12] Rehmani.S.,Sunitha.M.S.:Perfectgeodesicfuzzygraphs,International Journal of Pure and Applied Mathematics, Volume 120, No. 6 (2018), 6243-6251.

[13] Rehmani. S., Sunitha. M. S.:On the geodetic iteration number and geodeticnumberofafuzzygraphbasedonsumdistance,International Journal of Advanced Research in Science, Engineering and Technology, Vol.7, Issue 8, (2020)

[14] Rehmani. S.:Pseudo s-geodetic number of a fuzzy graph, London Journal of Engineering Research, London Journals Press, Volume 22, Issue 7, (2022).

[15] Stalin.D.:Pseudo Edge Geodetic Number and Perfect Edge Geodetic Graph, International Journal of Scientific & Engineering Research, Volume 6, Issue 3, (2015).

[16] Suvarna.N.T.,Sunitha.M.S.:Convexity and Types of Arcs& Nodes in Fuzzy Graphs, Scholar’s Press, (2015).

[17] Yeh.R.T.,Bang.S.Y.:Fuzzy relations, fuzzy graphs and their application to clustering analysis, In Fuzzy sets and their Application to Cognitive and Decision Processes, Zadeh L.A.,Fu.K.S.Shimura M.Eds, Academic Press, New York, (1975), 125-149.

[18] Zadeh.L.A.:Fuzzysets,InformationandControl,8(1965),338-353.