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RESEARCH ARTICLE

Old and New on some IBNR Methods

Werner Hürlimann

Independent Scholar, Feldstrasse 145, CH-8004 Zürich

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Corresponding Author

Werner Hürlimann

Abstract

Three maincategories of IBNR methods are considered: the standard IBNR methods (chain-ladder, Cape Cod, Bornhuetter-Ferguson), the IBNR loss ratio methods (loss ratios instead of link ratios) and the stochastic IBNR methods (calculation of standard deviation and percentiles). The main IBNR loss ratio methods include the individual and collective loss ratio methods as well as a credibility mixture of them, the so-called credibility loss ratio method. The latter includes variants by Benktander, Neuhaus and an optimal version by the author. We observe that the standard IBNR methods can be reinterpreted in the context of the IBNR loss ratio methods and extended to optimal credible standard IBNR methods. Among the stochastic IBNR methods we focus on stochastic chain-ladder models (including the multivariate setting), distribution based reserving models and multi-state reserving models. In particular, we propose a new and simple stochastic IBNR model based on the log-Laplace distribution. Numerical examples demonstrate that it is comparable in accuracy to the standard and loss ratio IBNR methods. As a new feature,the expected dynamic development of IBNR methods is studied in detail. It uses newly defined case reserve development factors. Numerical examples illustrate and compare the divers IBNR methods.

1 Introduction

On the level of a line of insurance business (LOB), an insurer thinks in terms of *origin periods*, *development periods* and *calendar periods*, which form the horizontal axis, vertical axis and diagonal of the loss triangle of past reported claims available for best estimation of ultimate aggregate paid claims (UAPC) and incurred but not reported (IBNR) claims reserves.

Assume that the start of the LOB has been at time $t=0$ and has lasted for n years up to the actual *analysis date* $t_0 = n$. For simplicity consider one-year insurance cover periods. In a $n \times n$ loss triangle, one has n origin periods (the different years to which claims are assigned) and n development periods (the years in which claims develop following each origin period). For origin periods based on claim occurrence (e.g. accident years) the premiums chosen should be the *earned premiums*, whereas for origins periods based on when the business was written (e.g. underwriting years), the premiums chosen should be *written premiums*. The choice of using an accident year or underwriting year classification will depend upon the accounting convention to be applied. For instance, accident year is the natural choice under US GAAP, where Lloyd’s accounting requires underwriting year (e.g. Boulter and Grubbs [4], p.11).

The loss triangle of (incremental) reported claims $\{S_{ik}\}$, incorporating for convenience the premiums of each accident year, can be displayed in the following form:

		Development period					
Origin period	**Premium**	**1**	**2**	**…**	**…**	**n-1**	**n**
1	P_1	S_{11}	S_{12}			S_{12}	S_{1n}
2	P_2	S_{21}	S_{22}	…	…	S_{2n-1}	
…	…	…	…	…	…		
…	…	…	…	…			
n-1	P_{n-1}	S_{n-11}	S_{n-12}				
n	P_n	S_{n1}					

The number $S_{ik}, 1 \le i,k \le n,$ represent the (incremental) *reported claims* (=paid claims + case reserve) for claims assigned to origin period i, which have beenreported in year $i+k-1$. At analysis date $t_0 = n$, each origin period $i \in \{1,2,...,n\}$ has reported claims for exactly $n-i+1$ development periods indexed with $k \in \{1,2,...,n-i+1\}$. The corresponding loss triangle of (cumulative) reported claims contains known entries defined by

$$C_{ik} = \sum_{j=1}^{k} S_{ij}, \quad i \in \{1,...,n\}, k \in \{1,2,...,n-i+1\}. \tag{1.1}$$

Reported claims for the lower triangle must be predicted from the upper triangle entries (1.1). The most important statistic $RC_i := C_{in-i+1}$ represents the most recent value of the (cumulative) reported claims corresponding to the origin period $i \in \{1,2,...,n\}$ and the current calendar period $k = n-i+1$ at analysis date $t_0 = n$, which is found in the diagonal of the loss triangle $\{C_{ik}\}$. Some important IBNR reserving methods are described by following a classification into 3 categories:

1. The *standard IBNR methods*, which consist of the commonly used reserving methods like the Chain-Ladder, the Cape Cod and the Bornhuetter-Ferguson methods (consult Boulter and Grubbs [4] for a useful elementary description).

2. The *IBNR loss ratio methods*, which consist of a modification of the standard IBNR methods based on loss ratios (average ratio of incremental reported claims to premiums for each development period) instead of link ratios (average ratio of cumulative reported claims between two consecutive development periods). Our presentation is inspired from Hürlimann [21], which is based on older ideas by Benktander [3], Neuhaus [38] and Mack [33].

3. The *stochastic IBNR methods*, which allow besides point estimates also a quantification of the standard deviation and percentiles of the IBNR reserves. Besides a short survey pointing out to the literature we introduce a novel log-Laplace IBNR model.

The simplest methods of the first two categories are specified in detail and made ready for direct implementation. The few chosen methods do not at all exhaust the vast amount of possibilities. In particular, all methods can also be applied to a loss triangle of (incremental) paid claims as well as extended to a combination of paid and reported loss triangles (Munich Chain-Ladder method by Quarg and Mack [44] for example). Finally, the field of IBNR methods is still in progress and the need for appropriate stochastic methods is increasing. Among recent work, one finds a handbook by Radtke and Schmidt [45], an extensive bibliography by Schmidt [53], and Ph.D. theses by Salzmann [50] and Happ [14].

2 Standard IBNR Methods

The standard IBNR methods include the three simplest and most common methods that are applied in practice. All these methods use the so-called *chain-ladder factors* defined by the average link ratios

$$f_k^{CL} = \sum_{i=1}^{n-k} C_{ik+1} / \sum_{i=1}^{n-k} C_{ik}, \quad k = 1,...,n-1. \tag{2.1}$$

From (2.1) one gets the (ultimate) loss development factors, called *LDF reported*,

$$F_k^r = \prod_{j=k}^{n-1} f_j^{CL}, \quad k = 1,...,n-1, \quad F_n^r = 1, \tag{2.2}$$

which represent the average ratio of the *ultimate aggregate paid claims* (UAPC) to the (cumulative) reported claims of each origin period after k years of development. From the LDF reported one gets immediately the *chain-ladder lag-factors*

$$p_i^{CL} = \frac{1}{F_{n-i+1}^r}, \quad i = 1,...,n, \tag{2.3}$$

representing the average ratio of UAPC from origin period i, which are reported at analysis date in the development period $n-i+1$, and the *chain-ladder IBNR factors*

$$q_i^{CL} = 1 - p_i^{CL}, \quad i = 1,...,n, \tag{2.4}$$

representing the average ratio of UAPC from origin period i, which remain unreported at analysis date in the development year $n-i+1$. In all of the following, the upper index $\bullet$ stands for the chosen method of calculation. For each origin period $i \in \{1,2,...,n\}$, we specify the UAPC at analysis date t_0, which is denoted by $U_i^\bullet$, and the IBNR reserve, which is denoted by $IBNR_i^\bullet$. The expected dynamic development is specified in Section 5.

The Chain-Ladder Method

$$U_i^{CL} = \frac{RC_i}{p_i^{CL}}, \quad IBNR_i^{CL} = q_i^{CL} \cdot U_i^{CL}, \quad i = 1,...,n \tag{2.5}$$

One notes that $p_1^{CL} = 1, q_1^{CL} = 0$, that is the claims assigned to the first origin period are fully developed at the analysis date, hence UAPC coincide with the reported claims and the IBNR reserve vanishes.

The Cape Cod Method

It uses an estimated average loss ratio (losses to weighted premiums) over all origin periods:

$$LR = \sum_{i=1}^{n} RC_i / \sum_{i=1}^{n} p_i^{CL} \cdot P_i \tag{2.6}$$

$$IBNR_i^{CC} = q_i^{CL} \cdot LR \cdot P_i,$$
$$U_i^{CC} = RC_i + IBNR_i^{CC} = p_i^{CL} \cdot U_i^{CL} + \left(1 - p_i^{CL}\right) \cdot LR \cdot P_i, \quad i = 1,\ldots,n \tag{2.7}$$

The ratio (2.6) is obtained by balancing UACP over all origin periods (e.g. Dahl [7], (9.5), p.12): $\sum_{i=1}^{n} U_i^{CC} = \sum_{i=1}^{n} RC_i + \sum_{i=1}^{n} IBNR_i^{CC} = \sum_{i=1}^{n} RC_i + LR \cdot \sum_{i=1}^{n} q_i^{CL} \cdot P_i = LR \cdot \sum_{i=1}^{n} P_i$

The Bornhuetter-Ferguson Method

Instead of an average loss ratio, it uses a selected initial loss ratio LR_i for each origin period:

$$IBNR_i^{BF} = q_i^{CL} \cdot LR_i \cdot P_i,$$
$$U_i^{BF} = RC_i + IBNR_i^{BF} = p_i^{CL} \cdot U_i^{CL} + \left(1 - p_i^{CL}\right) \cdot LR_i \cdot P_i, \quad i = 1,\ldots,n \tag{2.8}$$

3 IBNR Loss Ratio Methods

Instead of the link ratios (2.1), the analysis is based on loss ratios representing the incremental amount of reported claims per unit of premium in each development period, which are defined by

$$m_k = \sum_{i=1}^{n-k+1} S_{ik} \Big/ \sum_{i=1}^{n-k+1} P_i, \quad k \in \{1,\ldots,n\}. \tag{3.1}$$

Derived from the m_k's one gets the *loss ratio lag-factors*

$$p_i^{LR} = \sum_{k=1}^{n-i+1} m_k \Big/ \sum_{k=1}^{n} m_k, \quad i = 1,\ldots,n, \tag{3.2}$$

representing the average ratio of UAPC from origin period i, which are reported at analysis date in the development period $n-i+1$, and the *loss ratio IBNR factors*

$$q_i^{LR} = 1 - p_i^{LR}, \quad i = 1,\ldots,n, \tag{3.3}$$

representing the average ratio of UAPC from origin period i, which remain unreported at analysis date in the development period $n-i+1$.

We distinguish between three main IBNR loss ratio methods: the Individual LR method, the Collective LR method and the Credibility LR method, where the latter itself divides into the Benktander Credibility LR method, the Neuhaus Credibility LR method and the Optimal Credibility LR method.

Individual LR Method

As in the chain-ladder method, the UAPC of each origin period depends on the current *individual* claims experience at analysis date:

$$U_i^{ind} = \frac{RC_i}{p_i^{LR}}, \quad IBNR_i^{ind} = q_i^{LR} \cdot U_i^{ind}, \quad i = 1,\ldots,n \tag{3.4}$$

Collective LR Method

The UAPC of each origin period depends on the overall *collective* claims experience and the premium assigned to the origin period:

$$U_i^{coll} = P_i \cdot \sum_{k=1}^{n} m_k, \quad IBNR_i^{coll} = q_i^{LR} \cdot U_i^{coll}, \quad i = 1,...,n \tag{3.5}$$

The Individual and Collective LR methods correspond to extreme positions. The individual method considers the current (cumulative) reported claims RC_i as fully credible predictive for $IBNR_i$ and ignores the prior estimate U_i^{coll} while the collective method ignores RC_i and relies fully on U_i^{coll} to settle $IBNR_i$. It is natural to consider as compromise between these two positions a credibility mixture.

Credibility LR Method

$$\begin{aligned} U_i^{cred} &= Z_i \cdot U_i^{ind} + (1 - Z_i) \cdot U_i^{coll}, \\ IBNR_i^{cred} &= Z_i \cdot IBNR_i^{ind} + (1 - Z_i) \cdot IBNR_i^{coll}, \quad i = 1,...,n, \end{aligned} \tag{3.6}$$

where the coefficients Z_i represent credibility weights. These coefficients may be determined according to several proposals (author [21]):

<u>Benktander Credibility LR Method</u>

$$Z_i^{BC} = p_i^{LR}, \quad i = 1,...,n \tag{3.7}$$

<u>Neuhaus Credibility LR Method</u>

$$Z_i^{NC} = p_i^{LR} \cdot \sum_{k=1}^{n} m_k, \quad i = 1,...,n \tag{3.8}$$

<u>Optimal Credibility LR Method</u>

$$Z_i^{OC} = \frac{p_i^{LR}}{p_i^{LR} + \sqrt{p_i^{LR}}}, \quad i = 1,...,n \tag{3.9}$$

It is important to remark that the standard IBNR methods of Section 2 can be reinterpreted in the context of the IBNR loss ratio methods and extended to optimal credible standard IBNR methods as follows.

The Chain-Ladder Method

It is similar to the individual LR method, where the loss ratio lag-factors (3.2) are replaced by the chain-ladder lag-factors (2.3):

$$U_i^{ind} = \frac{RC_i}{p_i^{CL}}, \quad IBNR_i^{ind} = q_i^{CL} \cdot U_i^{ind}, \quad i = 1,...,n \tag{3.10}$$

The Cape Cod Method

It is a (Benktander type) credibility mixture of the type (3.6) with

$$U_i^{ind} = \frac{RC_i}{p_i^{CL}}, \quad U_i^{coll} = LR \cdot P_i, \quad Z_i = p_i^{CL}, \quad i = 1,...,n \tag{3.11}$$

Another credibility improvement of the Cape Cod method is found in Barnett [2].

The Optimal Cape Cod Method

It is the optimal credibility mixture of the type (3.6) with

$$U_i^{ind} = \frac{RC_i}{p_i^{CL}}, \quad U_i^{coll} = LR \cdot P_i, \quad Z_i = \frac{p_i^{CL}}{p_i^{CL} + \sqrt{p_i^{CL}}}, \quad i = 1,...,n \tag{3.12}$$

The Bornhuetter-Ferguson Method

It is a (Benktander type) credibility mixture of the type (3.6) with

$$U_i^{ind} = \frac{RC_i}{p_i^{CL}}, \quad U_i^{coll} = LR_i \cdot P_i, \quad Z_i = p_i^{CL}, \quad i = 1,...,n \tag{3.13}$$

Under suitable model assumptions, it is known that these credibility weights produce the optimal linear combination of the two predictors U_i^{ind} and U_i^{coll} of UAPC in the sense of minimum quadratic loss (e.g. Dahl [7], Theorem, p.10).

The Optimal Bornhuetter-Ferguson Method

It is the optimal credibility mixture of the type (3.6) with

$$U_i^{ind} = \frac{RC_i}{p_i^{CL}}, \quad U_i^{coll} = LR_i \cdot P_i, \quad Z_i = \frac{p_i^{CL}}{p_i^{CL} + \sqrt{p_i^{CL}}}, \quad i = 1,...,n \tag{3.14}$$

4 Stochastic IBNR Methods

We restrict ourselves to a short survey of available stochastic methods and refer to the literature for the more complex recent developments in this area. A recommendable textbook is Wüthrich and Merz[58].

Stochastic chain-ladder models

The standard chain-ladder method is the simplest and most suggestive tool in claims reserving. Various attempts have been made to justify it as a stochastic model (e.g. Mack and Venter [34]). Remarkable progress was achieved by Schnieper [55], and Mack [28]-[30], [32], who considered models involving assumptions on conditional distributions. The paper by Schmidt and Schnaus [54] extends the model of Mack and proposes a basic model in a decision theoretic setting. The model characterizes optimality of the chain ladder factors as predictors of non-observable development factors and hence optimality of the chain ladder predictors of aggregate claims at the end of the first non-observable calendar year. These authors also present a model in which the chain ladder predictor of ultimate aggregate claims turns out to be unbiased. However, Taylor [57] has shown that the chain ladder forecast is upward biased in general.
More recent developments concern the extension of the chain-ladder technique to a multivariate setting. Since Ajne [1] it is well-known that the univariate chain-ladder method cannot be applied to a portfolio of risks consisting of several sub-portfolios. In general, the chain-ladder predictors of sums differ from the sums of chain-ladder

predictors. This is due to the fact that the univariate chain-ladder method neglects the dependence structure between the sub-portfolios of a portfolio. Braun [6] has constructed a bivariate model, which allows calculation of prediction errors for the sum of the univariate chain-ladder predictors taking into account the correlation structure of sub-portfolios. In the recent paper by Pröhl and Schmidt [42], the authors extend both the method of Braun and the univariate model of Schmidt and Schnaus to a multivariate chain-ladder model. In particular, this model resolves the problem of non-additivity of the univariate chain-ladder method by arguing that the only reasonable predictor of the sum of non-observable (future) aggregate claims of a portfolio is the multivariate chain-ladder predictor consisting of the sum of the multivariate chain-ladder predictors of the sub-portfolios. The author [19] proposes a linear approximate estimation of the bivariate chain-ladder factors, which leads to simple approximate lower and upper bounds for the IBNR claims reserve of a portfolio and its components. It is based on a Taylor approximation, where the second order quadratic term can be neglected in practice.

Finally, the application of the chain-ladder technique to reinsurance must be done with care. The analysis of reinsurance treaties often requires a separate analysis of both paid and reported (or incurred) claims. However, such a method can lead to very different ultimate projections. Quarg and Mack [44] proposed a solution to this well-known problemin their presentation of the so-called Munich Chain-Ladder method (see also Merz and Wüthrich [36]). In reinsurance it is also advisable to separate attritional claims from large claims in loss triangles. A technique for this is found in Klemmt [26].

Distribution based reserving models

A main and very important advantage of the standard and loss ratio IBNR methods is their distribution-free validity. However, in a risk consulting environment, there is an accrued interest to know more about the standard deviation and the higher percentile values. Therefore, attempts to model adequately not only the mean of the IBNR claims reserves but also its full distribution have the potential to retain more attention from both a theoretical and practical viewpoint. Early developments in this area include work by Bühlmann et al. [5], and Hertig [15].

Later on, Mack has proposed distribution dependent IBNR claims reserving methods, in particular a cross-classified parametric method of multiplicative type based on the gamma distribution (see Mack [31], Section 3.3.3, pp. 281-283). The author [20] considers two modifications of this latter model. The first model assumes independent development periods and allocates the coefficient of variation of the total ultimate claims of a line of business with multiple origin periods to the coefficient of variation of the total ultimate claims of a single origin period inverse proportionally to the squared-root premium volumes. The second model extension is based on a simple Fréchet like multivariate distribution, which models the whole range of dependence between independent and comonotone dependent development periods. The chosen model uses only one additional dependence parameter, which is chosen such that it yields the most conservative model for IBNR claims reserving with respect to the concordance order for the bivariate margins of this model.

In the next Section 5, we propose a new and simple stochastic IBNR model based on the log-Laplace distribution. The numerical examples of Section 6 demonstrate that it is comparable in accuracy to the standard and loss ratio IBNR methods summarized in the Sections 2 and 3.

Multi-state reserving models

The paper by Orr [40] shows how a simple multi-state claims number reserving model can be obtained from a multi-state Markov chain modelling of the claims reserving process along the line of papers including Hachemeister [13], Norberg [39] and Hesselager [16]. The author[25] considers the simplest extension of the model of Orr [40] to a multi-state aggregate claims reserving model with gamma distributed claim sizes, which can be approximated for sufficiently large portfolios by a gamma distributed aggregate claims reserving model.

5 The log-Laplace IBNR model

In recent years the log-Laplace distribution has become quite popular in applied stochastic modeling. There exist several theoretical reasons for this, in particular the power tails and the self-similarity of this law. Kozubowski and Podgorski [27] provide an excellent overview of the historical development and applications of the log-Laplace models together with new results on their properties with particular attention to their stability with respect to

geometric products. To our knowledge the application of the log-Laplace distribution to the important IBNR claims reserving problem has not been considered so far.
We consider the three parameter "skew log-Laplace" distribution as defined in [27] and also called "double Pareto" in Reed [46]. A random variable X distributed according to the density (5.1) below is said to have a log-Laplace distribution with parameters $\delta > 0, \alpha > 0, \beta > 0$. This distribution is denoted by $LL(\delta, \alpha, \beta)$. Analytical expressions for the distribution function and percent point function are given in (5.2).

$$f_X(x) = \begin{cases} \frac{1}{\delta} \frac{\alpha\beta}{\alpha+\beta} \left(\frac{x}{\delta}\right)^{\beta-1}, & 0 < x < \delta \\ \frac{1}{\delta} \frac{\alpha\beta}{\alpha+\beta} \left(\frac{\delta}{x}\right)^{\alpha+1}, & x \geq \delta \end{cases} \tag{5.1}$$

$$F_X(x) = \begin{cases} \frac{\alpha}{\alpha+\beta} \left(\frac{x}{\delta}\right)^{\beta}, & 0 < x < \delta \\ 1 - \frac{\beta}{\alpha+\beta} \left(\frac{\delta}{x}\right)^{\alpha}, & x \geq \delta \end{cases}, \quad F_X^{-1}(u) = \begin{cases} \delta \cdot \left(\frac{\alpha+\beta}{\alpha} u\right)^{\frac{1}{\beta}}, & u \in \left(0, \frac{\alpha}{\alpha+\beta}\right] \\ \delta \cdot \left(\frac{\alpha+\beta}{\beta}(1-u)\right)^{-\frac{1}{\alpha}}, & u \in \left(\frac{\alpha}{\alpha+\beta}, 1\right) \end{cases} \tag{5.2}$$

In practice, the "symmetric log-Laplace" is even more frequently encountered. It is the "symmetric" version $\alpha = \beta = \frac{1}{b}, \delta = e^{\mu}$ of the three parameter "skew log-Laplace" with density, distribution function and percent point function given by

$$f_X(x) = \begin{cases} \frac{1}{2} e^{-\mu} \left(e^{-\mu} x\right)^{\frac{1}{b}-1}, & 0 < x < e^{\mu} \\ \frac{1}{2} e^{-\mu} \left(e^{-\mu} x\right)^{-\frac{1}{b}-1}, & x \geq e^{\mu} \end{cases} \tag{5.3}$$

$$F_X(x) = \begin{cases} \frac{1}{2} \left(e^{-\mu} x\right)^{\frac{1}{b}}, & 0 < x < e^{\mu} \\ 1 - \frac{1}{2} \left(e^{-\mu} x\right)^{-\frac{1}{b}}, & x \geq e^{\mu} \end{cases}, \quad F_X^{-1}(u) = \begin{cases} e^{\mu} \cdot (2u)^{-\frac{1}{b}}, & u \in \left(0, \frac{1}{2}\right] \\ e^{\mu} \cdot (2-2u)^{\frac{1}{b}}, & u \in \left(\frac{1}{2}, 1\right) \end{cases} \tag{5.4}$$

It is interesting to observe that the symmetric log-Laplace has Pareto tails with index $\frac{1}{b}$, and thus this simple special model is consistent in the tail region with Mandelbrot's Pareto hypothesis for financial returns (see Mandelbrot [35], Fama [11], [12]. In particular, the mean excess function is linear and increasing in the tails, and in accordance with extreme value theory (e.g. Embrechts et al. [9]), the symmetric log-Laplace is thus susceptible to model long-tailed insurance claims data. The symmetric log-Laplace can be viewed as special case of the logarithmic double Weibull distribution (e.g. Hürlimann [17], Section 4).
The limiting cases of the three parameter log-Laplace as $\alpha \to \infty$ or $\beta \to \infty$ are allowed (Pareto type tails at zero and infinity respectively). In case $\alpha \to \infty$ one obtains a special beta distribution with density

$$f_X(x) = \begin{cases} \frac{\beta}{\delta} \left(\frac{x}{\delta}\right)^{\beta-1}, & 0 < x < \delta \\ 0, & x \geq \delta \end{cases} \tag{5.5}$$

and in case $\beta \to \infty$ one gets the Pareto distribution with density

$$f_X(x) = \begin{cases} 0, & 0 < x < \delta \\ \frac{\alpha}{\delta} \left(\frac{\delta}{x}\right)^{\alpha+1}, & x \geq \delta \end{cases} \tag{5.6}$$

The moments of order k of the log-Laplace distribution, which exist only under parameter restriction, are given by

$$m_k = E\left[X^k\right] = \delta^k \cdot \frac{\alpha\beta}{(\alpha - k)(\beta + k)}, \quad \alpha > k. \tag{5.7}$$

A log-Laplace IBNR claims reserving model can be obtained as follows. We assume that the reported claims $\{C_{ik}\}_{1 \le i,k \le n}$ follow independent random variables with log-Laplace distributions $LL(\delta_k, \alpha_i, \beta_i)$. The parameters must be predicted from the upper loss triangle $C_{ik}, i \in \{1,...,n\}, k \in \{1,2,...,n-i+1\}$. A comparison of the ratio of mean values using (5.7) suggests a natural estimation of the δ_k parameters using the chain-ladder factors (2.1) as average link ratios between $E[C_{ik+1}]$ and $E[C_{ik}]$ as follows:

$$\hat{\delta}_k = \frac{\hat{\delta}_{k+1}}{f_k^{CL}}, \quad k = 1,...,n-1. \tag{5.8}$$

The estimation of δ_n can be done according to some optimality criterion. We suggest to choose $\hat{\delta}_n$ such that the overall estimated IBNR claims reserve $\sum_{i=1}^{n} \widehat{IBNR}_i$ is closest to the value obtained from the optimal credibility method in Section 3 (see the numerical examples in Section 6). Assuming now that the δ_k parameters are known, we estimate the parameters α_i, β_i using the method of maximum likelihood estimation. Under the assumption that the observed reported claims are independent, the log-likelihood function equals

$$\ln(L) = \sum_{i=1}^{n} \sum_{k=1}^{n-i+1} \left\{ \begin{array}{l} I(C_{ik} < \delta_k) \cdot \left[(\beta_i - 1)\ln(C_{ik}) + \ln\left(\frac{\alpha_i \beta_i}{\alpha_i + \beta_i}\right) - \beta_i \ln(\delta_k) \right] \\ + I(C_{ik} \ge \delta_k) \cdot \left[-(\alpha_i + 1)\ln(C_{ik}) + \ln\left(\frac{\alpha_i \beta_i}{\alpha_i + \beta_i}\right) + \alpha_i \ln(\delta_k) \right] \end{array} \right\}. \tag{5.9}$$

The maximum likelihood estimators of the parameters $\alpha_i, \beta_i, \ 1 \le i,k \le n,$ are those values, which maximize $\ln(L)$, that is the simultaneous solutions of the equations

$$0 = \partial \ln(L) / \partial \alpha_i = (n-i+1) \cdot \left(\frac{1}{\alpha_i} - \frac{1}{\alpha_i + \beta_i} \right) + \sum_{k=1}^{n-i+1} I(C_{ik} \ge \delta_k) \cdot \ln\left(\frac{\delta_k}{C_{ik}}\right), \quad i = 1,...,n, \tag{5.10}$$

$$0 = \partial \ln(L) / \partial \beta_i = (n-i+1) \cdot \left(\frac{1}{\beta_i} - \frac{1}{\alpha_i + \beta_i} \right) + \sum_{k=1}^{n-i+1} I(C_{ik} < \delta_k) \cdot \ln\left(\frac{C_{ik}}{\delta_k}\right), \quad i = 1,...,n. \tag{5.11}$$

For convenience consider the quantities

$$A_i = \frac{1}{n-i+1} \ln\left\{ \prod_{k=1}^{n-i+1} \max\left(\frac{C_{ik}}{\delta_k}, 1\right) \right\}, \quad B_i = \frac{1}{n-i+1} \ln\left\{ \prod_{k=1}^{n-i+1} \max\left(\frac{\delta_k}{C_{ik}}, 1\right) \right\}, \quad i = 1,...,n \tag{5.12}$$

Then the equations (5.10) and (5.11) are equivalent to the equations

$$\frac{\beta_i}{\alpha_i(\alpha_i+\beta_i)} = A_i, \quad \frac{\alpha_i}{\alpha_i(\alpha_i+\beta_i)} = B_i, \quad i = 1,...,n. \tag{5.13}$$

Solving (5.13) yields the maximum likelihood estimators

$$\hat{\alpha}_i = \frac{1}{A_i + \sqrt{A_i B_i}}, \quad \hat{\beta}_i = \frac{1}{B_i + \sqrt{A_i B_i}}, \quad i = 1,...,n. \tag{5.14}$$

Of course, the cases $A_i = 0$ or $B_i = 0$ correspond to the limiting cases $\hat{\alpha}_i \to \infty$ or $\hat{\beta}_i \to \infty$ defined under (5.5) and (5.6). In our stochastic model the random IBNR claims reserves are obtained as the independent differences defined by

$$IBNR_i = C_{in} - C_{in-i+1}, \quad i = 1,...,n \tag{5.15}$$

The mean and variance of the IBNR claims reserves are obtained from the formulas

$$E[IBNR_i] = E[C_{in} - C_{in-i+1}] = (\delta_n - \delta_{n-i+1})\frac{\alpha_i \beta_i}{(\alpha_i - 1)(\beta_i + 1)}, \quad i = 1,...,n, \tag{5.16}$$

$$\begin{aligned} Var[IBNR_i] &= Var[C_{in}] + Var[C_{in-i+1}] \\ &= (\delta_n^2 + \delta_{n-i+1}^2)\left[\frac{\alpha_i \beta_i}{(\alpha_i - 2)(\beta_i + 2)} - \frac{(\alpha_i \beta_i)^2}{(\alpha_i - 1)^2 (\beta_i + 1)^2}\right], \quad i = 1,...,n, \end{aligned} \tag{5.17}$$

Percentiles of the IBNR claims reserves can also be obtained. Unfortunately, the distribution of independent differences of log-Laplace distributions is not available in analytical closed-form. Instead, various approximations can be used. For example, it is possible to evaluate a distribution free Chebyshev-Markov upper bound of these percentiles, which is based on the mean, variance, skewness and kurtosis (use Theorem 4.1 in Hürlimann [18]). Alternatively, and as suggested by the technical specification QIS5 [43] for Solvency II, a practical log-normal approximation of percentiles based on the mean and variance might be appropriate. Setting $\mu_i = E[IBNR_i], \sigma_i^2 = Var[IBNR_i], i = 1,...,n,$ the log-normal IBNR value-at-risk approximation to the confidence level $1-\varepsilon$ is given by ($\Phi^{-1}(u)$ denotes the u-percentile of the standard normal distribution)

$$VaR_{1-\varepsilon}[IBNR_i] = \mu_i \cdot \exp\left[\Phi^{-1}(1-\varepsilon)\sqrt{\ln\left(1 + \left[\tfrac{\sigma_i}{\mu_i}\right]^2\right)}\right] / \sqrt{1 + \left[\tfrac{\sigma_i}{\mu_i}\right]^2}, \quad i = 1,...,n. \tag{5.18}$$

6 Expected Dynamic Reserve Development

The upper index $\bullet$ stands for the chosen IBNR method. For each origin period $i \in \{1,2,...,n\}$, the UACP at analysis date t_0 is denoted by $U_i^\bullet$. The (average) dynamic development of IBNR reserve and (cumulative) reported claims of an origin period for the development years following the analysis date is described by the time dependent formulas

$$IBNR_i^\bullet(t_0 + k) = q_{i-k}^\bullet \cdot U_i^\bullet, \quad k = 1,...,i-1, \ i = 1,...,n, \tag{6.1}$$

$$RC_i^\bullet(t_0 + k) = U_i^\bullet - IBNR_i^\bullet(t_0 + k), \quad k = 1,...,i-1, \ i = 1,...,n. \tag{6.2}$$

For each origin period the known (cumulative) reported claims RC_i at analysis date are obtained as the sum of the known (cumulative) paid claims P_i and the known (outstanding) case reserves R_i, that is

$$RC_i = P_i + R_i, \quad i = 1,...,n. \tag{6.3}$$

Assume that case reserves, after the analysis date, develop according to *case reserve development factors* $CRDF = (f_1,..., f_n)$ such that $0 < f_k \le 1, k = 1,..., m < n, \ f_k = 0, k = m+1,..., n$, and set by convention $f_0 = 1$. We suppose that the development of case reserves for claims reported after the analysis date follow the same deterministic pattern $CRDF = (f_1,..., f_n)$. Through analysis one gets the following formulas for the case reserves and (cumulative) paid claims of an origin period $i \in \{1,2,..., n\}$ after the analysis date:

$$\begin{aligned} R_i^\bullet(t_0+k) &= \left(\prod_{s=1}^{k} f_s\right)\cdot R_i \\ &+ \sum_{j=1}^{k}\left(\prod_{s=0}^{k-j} f_s\right)\cdot\left(RC_i^\bullet(t_0+j) - RC_i^\bullet(t_0+j-1)\right), \quad k = 1,..., i+m, \end{aligned} \tag{6.4}$$

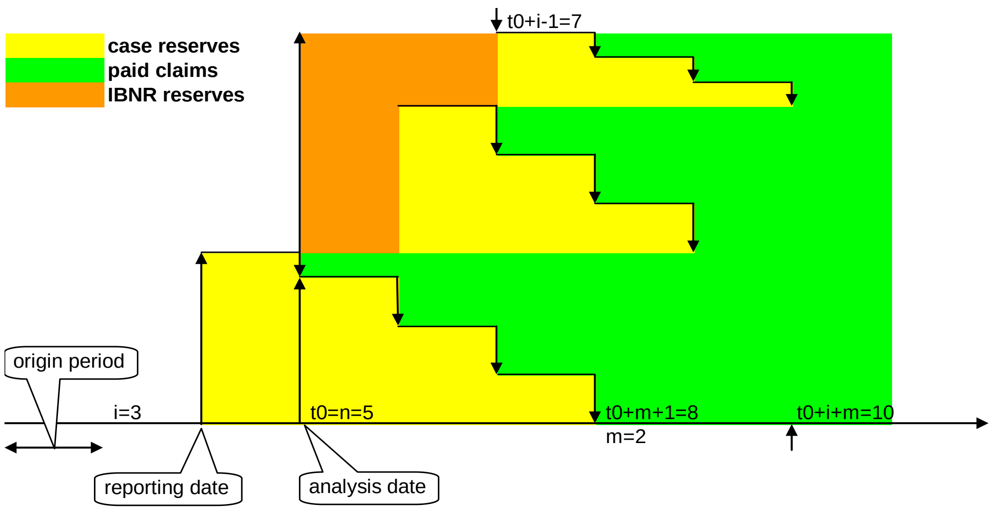
$$P_i^\bullet(t_0+k) = RC_i^\bullet(t_0+k) - R_i^\bullet(t_0+k), \quad k = 1,..., i+m, \tag{6.5}$$

where by convention $RC_i^\bullet(t_0+j) = U_i^\bullet, \quad j = i,..., i+m$. Formula (5.4) shows that the case reserves at date $t_0 + k$ are obtained as the sum of the (remaining) case reserves known from the analysis date and all those (remaining) case reserves emanating from incremental reported claims of the development periods $(t_0 + j - 1, t_0 + j], \ j = 1,..., k,$ which are revealed after the analysis date. Using (6.1) and (6.2) one checks without difficulty that at date $t_0 + i + m$ all claims from all origin periods have been reported and paid out, that is

$$R_i^\bullet(t_0+i+m) = 0, \quad P_i^\bullet(t_0+i+m) = RC_i^\bullet(t_0+i+m) = U_i^\bullet, \quad i = 1,..., n. \tag{6.6}$$

The Figure 6.1 illustrates graphically the expected dynamic reserve development process in a special case.

Figure 6.1: Expected dynamic reserving process for the claims of a specific origin period



7 Numerical example

The IBNR methods of Sections 2, 3 and 5 are illustrated with two different loss triangles. The first (fictive) loss triangle is taken from Boulter and Grubbs [4]. Its use is justified as far as it allows an easy understanding of the methods from simple figures. The second real-world loss triangle from A.M. Best concerns the Private Passenger Auto Liability insurance business. The reader is recommended to implement its own spreadsheet calculations in order to figure out the details.

Swiss Re exemplary loss triangle (Boulter and Grubbs [4])

Starting point is the following loss triangle of (cumulative) reported claims, premiums, chain-ladder factors and ultimate loss development factors (LDF reported):

	Development Period in Months						estimated
Origin Period	12	24	36	48	60	72	Premium
1995	90	210	310	420	500	500	625
1996	130	280	360	460	600		625
1997	140	290	440	600			625
1998	160	240	420				625
1999	120	260					625
2000	110						625
Chain-ladder factors	2.000	1.500	1.333	1.250	1.000	1.000	
LDF reported	5.000	2.500	1.667	1.250	1.000	1.000	

As input for the Bornhuetter-Ferguson method, it is necessary to estimate initial loss ratios for each origin period. As natural projection beyond the current calendar year 2000 we use average loss ratios:

	Development Period in Months						estimated
Origin Period	12	24	36	48	60	72	Loss Ratio
1995	0.144	0.192	0.160	0.176	0.128	0.000	0.800
1996	0.208	0.240	0.128	0.160	0.224	*0.000*	0.960
1997	0.224	0.240	0.240	0.256	*0.176*	*0.000*	1.136
1998	0.256	0.128	0.288	*0.197*	*0.176*	*0.000*	1.045
1999	0.192	0.224	*0.204*	*0.197*	*0.176*	*0.000*	0.993
2000	0.176	*0.205*	*0.204*	*0.197*	*0.176*	*0.000*	0.958
						Average	0.982

From the LDF reported one gets immediately the chain-ladder lag-factors (2.3) and IBNR factors (2.4):

	lag factors	IBNR factors
origin period i	p_i	q_i
1	1.00000	0.00000
2	1.00000	0.00000
3	0.80000	0.20000
4	0.60000	0.40000
5	0.40000	0.60000
6	0.20000	0.80000

Applying the standard IBNR methods of Section 2 one gets the estimates of UACP and IBNR reserves shown below, where the relative deviation is calculated with respect to the values obtained from the optimal credibility method in Section 3, which are summarized afterwards.

Standard Method:	Chain-Ladder		Cape Cod		Bornhuetter-Ferguson	
origin period i	U_i	IBNR_i	U_i	IBNR_i	U_i	IBNR_i
1	500	0	500	0	500	0
2	600	0	600	0	600	0
3	750	150	723	123	742	142
4	700	280	666	246	681	261
5	650	390	628	368	633	373
6	550	440	601	491	589	479
all origin periods	**3'750**	**1'260**	**3'718**	**1'228**	**3'745**	**1'255**
relative deviation	1.718%	5.060%	0.841%	2.364%	1.580%	4.635%

The next table displays the required loss ratios (3.1), loss ratio lag-factors (3.2) and loss ratio IBNR factors (3.3) as well as the credibility weights (3.7)-(3.9):

vector of m_k's	20.000%	20.480%	20.400%	19.733%	17.600%	0.000%	98.213%
origin period i	p_i	q_i	RC_i	P_i	ZB_i	ZN_i	ZO_i
1	1.00000	0.00000	500	625	1.00000	0.98213	0.50000
2	1.00000	0.00000	600	625	1.00000	0.98213	0.50000
3	0.82080	0.17920	600	625	0.82080	0.80613	0.47534
4	0.61988	0.38012	420	625	0.61988	0.60880	0.44050
5	0.41216	0.58784	260	625	0.41216	0.40480	0.39099
6	0.20364	0.79636	110	625	0.20364	0.20000	0.31095

The application of the IBNR loss ratio methods of Section 3 yields the following table, where again the relative deviation is calculated with respect to the optimal credibility method.

LR Method:	Individual Method		Collective Method		Benktander Method		Neuhaus Method		Optimal Method	
origin period i	U_i	IBNR_i	U_i	IBNR_i	U_i	IBNR_i	U_i	IBNR_i	U_i	IBNR_i
1	500	0	614	0	500	0	502	0	557	0
2	600	0	614	0	600	0	600	0	607	0
3	731	131	614	110	710	127	708	127	670	120
4	678	258	614	233	653	248	653	248	642	244
5	631	371	614	361	621	365	621	365	620	365
6	540	430	614	489	599	477	599	477	591	471
all origin periods	3'680	1'190	3'683	1'193	3'683	1'217	3'683	1'217	3'687	1'199
relative deviation	-0.193%	-0.815%	-0.099%	-0.526%	-0.099%	1.510%	-0.099%	1.473%	0.000%	0.000%

As next we use the log-Laplace IBNR method of Section 5. The δ_k parameters (5.1), the suggested estimate δ_n, and the other parameters in (5.14) and (5.16) are the following ones (a zero entry for α_i, β_i corresponds to a limiting case as the parameter goes to ∞):

index i / k	1	2	3	4	5	6
A_i	0.00000	0.04684	0.19180	0.14728	0.04002	0.00000
B_i	0.17815	0.00851	0.00000	0.00000	0.00000	0.08701
α_i	0.000	14.969	5.214	6.790	24.987	0.000
β_i	5.613	35.113	0.000	0.000	0.000	11.493
δ_k	120	240	360	480	600	600

The following table displays reported claims, ultimate aggregate paid claims and IBNR reserves by origin period using (5.18) as well as the relative deviations with respect to the optimal credibility method:

RC_i	U_i	IBNR_i
500	500	0
600	600	0
600	748	148
420	701	281
260	635	375
110	552	442
2'490	3'737	1'247
relative deviation	1.279%	2.930%

It is remarkable that all used IBNR methods yield stable values of the total UACP over all origin periods with relative deviations within -0.3% (individual loss ratio method) and 1.6% (chain-ladder method) of the chosen optimal credibility value. The total IBNR reserve over all origin periods is less stable with a range of relative deviation between -1.8% (individual loss ratio method) and 4% (chain-ladder method). The log-normal IBNR value-at-risk approximations (5.20) to the 90% confidence level together with the required standard deviations (5.19) are shown below. One notes that the total 90% value-at-risk 1'614 is 33% above the average optimal value 1'211.

90% Perc.	σ[IBNR_i]
0	0
0	0
333	232
465	144
412	28
501	45
1'614	**278**

A.M. Best loss triangle for the Private Passenger Auto Liability line of business

It is natural to ask whether the above numerical stable results are obtained by chance or whether they apply also to real-world loss triangles. Starting point is the following loss triangle:

Origin Period	Development Period in Months 12	24	36	48	60	72	84	96	108	120	estimated Premium
1994	35'676'056	41'669'881	43'323'363	43'984'520	44'150'389	44'190'053	44'221'618	44'253'092	44'258'956	44'268'181	60'910'084
1995	36'359'147	42'299'117	44'144'506	45'012'422	45'218'221	45'333'208	45'439'196	45'464'921	45'484'323		63'201'025
1996	37'061'077	42'961'817	45'102'738	45'941'840	46'334'591	46'540'489	46'671'487	46'709'735			65'433'383
1997	37'174'474	43'067'460	45'181'515	46'272'564	46'732'712	46'921'079	46'930'329				66'558'884
1998	37'412'178	43'605'555	46'018'215	47'237'090	47'784'254	47'981'868					67'498'626
1999	39'238'795	46'375'818	49'208'459	50'542'496	51'050'486						74'149'663
2000	41'434'635	49'562'321	52'535'619	53'952'933							80'259'593
2001	42'716'488	50'947'581	53'999'486								83'271'830
2002	45'093'972	53'343'255									87'467'542
2003	45'291'039										87'110'732
Chain-ladder factors	1.175	1.053	1.023	1.008	1.003	1.002	1.001	1.000	1.000	1.000	
LDF reported	1.283	1.092	1.037	1.014	1.006	1.003	1.001	1.000	1.000	1.000	

Initial loss ratios for the Bornhuetter-Ferguson method are obtained from the following table:

Origin Period	Development Period in Months 12	24	36	48	60	72	84	96	108	120	estimated Loss Ratio
1994	0.586	0.098	0.027	0.011	0.003	0.001	0.001	0.001	0.000	0.000	0.727
1995	0.575	0.094	0.029	0.014	0.003	0.002	0.002	0.000	0.000	*0.000*	0.720
1996	0.566	0.090	0.033	0.013	0.006	0.003	0.002	0.001	*0.000*	*0.000*	0.714
1997	0.559	0.089	0.032	0.016	0.007	0.003	0.000	*0.001*	*0.000*	*0.000*	0.707
1998	0.554	0.092	0.036	0.018	0.008	0.003	*0.001*	*0.001*	*0.000*	*0.000*	0.713
1999	0.529	0.096	0.038	0.018	0.007	*0.003*	*0.001*	*0.001*	*0.000*	*0.000*	0.694
2000	0.516	0.101	0.037	0.018	*0.007*	*0.003*	*0.001*	*0.001*	*0.000*	*0.000*	0.685
2001	0.513	0.099	0.037	*0.018*	*0.007*	*0.003*	*0.001*	*0.001*	*0.000*	*0.000*	0.679
2002	0.516	0.094	*0.037*	*0.018*	*0.007*	*0.003*	*0.001*	*0.001*	*0.000*	*0.000*	0.677
2003	0.520	*0.100*	*0.037*	*0.018*	*0.007*	*0.003*	*0.001*	*0.001*	*0.000*	*0.000*	0.687
										Average	0.700

Next we list the chain-ladder lag-factors (2.3) and IBNR factors (2.4):

	lag factors	IBNR factors
origin period i	p_i	q_i
1	1.00000	0.00000
2	0.99979	0.00021
3	0.99951	0.00049
4	0.99881	0.00119
5	0.99730	0.00270
6	0.99407	0.00593
7	0.98602	0.01398
8	0.96401	0.03599
9	0.91569	0.08431
10	0.77924	0.22076

The standard IBNR methods of Section 2 yield the following values:

Standard Method:	Chain-Ladder		Cape Cod		Bornhuetter-Ferguson	
origin period i	U_i	IBNR_i	U_i	IBNR_i	U_i	IBNR_i
1	44'268'181	0	44'268'181	0	44'268'181	0
2	45'493'803	9'480	45'493'546	9'223	45'493'803	9'480
3	46'732'628	22'893	46'732'182	22'447	46'732'631	22'896
4	46'986'202	55'873	46'985'757	55'428	46'986'251	55'922
5	48'111'924	130'056	48'109'647	127'779	48'112'021	130'153
6	51'354'849	304'363	51'358'242	307'756	51'355'444	304'958
7	54'718'095	765'162	54'738'904	785'971	54'721'377	768'444
8	56'015'256	2'015'770	56'098'039	2'098'553	56'033'989	2'034'503
9	58'254'733	4'911'478	58'507'598	5'164'343	58'338'112	4'994'857
10	58'121'989	12'830'950	58'758'234	13'467'195	58'509'747	13'218'708
all origin periods	**510'057'660**	**21'046'025**	**511'050'330**	**22'038'695**	**510'551'557**	**21'539'922**
relative deviation	-0.154%	-3.200%	0.041%	1.366%	-0.057%	-0.928%

The required input for application of the IBNR loss ratio methods is as follows:

vector of m_k's	54.013%	9.505%	3.389%	1.554%	0.573%	0.231%	0.108%	0.050%	0.020%	0.015%	**69.460%**
origin period i	p_i	q_i	RC_i	P_i	ZB_i	ZN_i	ZO_i				
1	1.00000	0.00000	44'268'181	60'910'084	1.00000	0.69460	0.50000				
2	0.99978	0.00022	45'484'323	63'201'025	0.99978	0.69445	0.49997				
3	0.99949	0.00051	46'709'735	65'433'383	0.99949	0.69424	0.49994				
4	0.99876	0.00124	46'930'329	66'558'884	0.99876	0.69374	0.49985				
5	0.99720	0.00280	47'981'868	67'498'626	0.99720	0.69265	0.49965				
6	0.99388	0.00612	51'050'486	74'149'663	0.99388	0.69035	0.49923				
7	0.98563	0.01437	53'952'933	80'259'593	0.98563	0.68462	0.49819				
8	0.96325	0.03675	53'999'486	83'271'830	0.96325	0.66907	0.49532				
9	0.91446	0.08554	53'343'255	87'467'542	0.91446	0.63518	0.48882				
10	0.77761	0.22239	45'291'039	87'110'732	0.77761	0.54013	0.46860				

The application of the IBNR loss ratio methods of Section 3 leads to the following results:

LR Method:	Individual Method		Collective Method		Benktander Method		Neuhaus Method		Optimal Method	
origin period i	U_i	IBNR_i	U_i	IBNR_i	U_i	IBNR_i	U_i	IBNR_i	U_i	IBNR_i
1	44'268'181	0	42'308'024	0	44'268'181	0	43'669'545	0	43'288'103	0
2	45'494'243	9'920	43'899'308	9'572	45'493'895	9'920	45'006'905	9'813	44'696'732	9'746
3	46'733'622	23'887	45'449'899	23'231	46'732'966	23'887	46'341'115	23'686	46'091'678	23'559
4	46'988'411	58'082	46'231'670	57'147	46'987'476	58'081	46'756'651	57'796	46'609'924	57'614
5	48'116'486	134'618	46'884'413	131'171	48'113'039	134'608	47'737'814	133'558	47'500'018	132'893
6	51'364'787	314'301	51'504'210	315'155	51'365'641	314'307	51'407'960	314'566	51'434'606	314'729
7	54'739'572	786'639	55'748'156	801'133	54'754'066	786'847	55'057'663	791'210	55'245'689	793'912
8	56'059'487	2'060'001	57'840'449	2'125'446	56'124'932	2'062'406	56'648'854	2'081'658	56'958'303	2'093'030
9	58'333'310	4'990'055	60'754'783	5'197'197	58'540'452	5'007'775	59'216'713	5'065'625	59'571'110	5'095'941
10	58'243'940	12'952'901	60'506'944	13'456'171	58'747'210	13'064'823	59'284'636	13'184'342	59'446'500	13'220'339
all origin periods	**510'342'038**	**21'330'403**	**511'127'856**	**22'116'221**	**511'127'856**	**21'462'653**	**511'127'856**	**21'662'254**	**510'842'661**	**21'741'762**
relative deviation	-0.098%	-1.892%	0.056%	1.722%	0.056%	-1.284%	0.056%	-0.366%	0.000%	0.000%

Our input for the log-Laplace IBNR method is

index i / k	1	2	3	4	5	6	7	8	9	10
A_i	0.00000	0.00000	0.00000	0.00000	0.00028	0.05821	0.11905	0.14392	0.18925	0.19029
B_i	0.07314	0.04918	0.02587	0.02035	0.00323	0.00000	0.00000	0.00000	0.00000	0.00000
α_i	0.000	0.000	0.000	0.000	816.650	17.178	8.400	6.948	5.284	5.255
β_i	13.673	20.332	38.658	49.131	239.550	0.000	0.000	0.000	0.000	0.000
δ_k	37'442'904	43'999'324	46'321'329	47'378'554	47'765'699	47'920'588	47'993'339	48'026'940	48'040'465	48'050'478

and it yields the following results

RC_i	U_i	IBNR_i
44'268'181	44'268'181	0
45'484'323	45'493'867	9'544
46'709'735	46'732'680	22'945
46'930'329	46'986'328	55'999
47'981'868	48'111'377	129'509
51'050'486	51'352'868	302'382
53'952'933	54'715'657	762'724
53'999'486	56'019'333	2'019'847
53'343'255	58'340'067	4'996'812
45'291'039	58'391'531	13'100'492
489'011'635	**510'411'887**	**21'400'252**
relative deviation	-0.067%	0.000%

All used IBNR methods yield remarkable stable values of the total UAPC over all origin periods with relative deviations within -0.14% (chain-ladder method) and 0.07% (collective loss ratio method) of the chosen optimal credibility value. The total IBNR reserve over all origin periods is a bit less stable with a range of relative deviation between -1.7% (chain-ladder method) and 3.3% (collective loss ratio method). Moreover, the chosen parameters of the log-Laplace method almost reproduce the results of the optimal credibility method. The log-normal IBNR value-at-risk approximations (5.20) to the 80% confidence level together with the required standard deviations (5.19) are shown below. One notes that the total 80% value-at-risk 29’617’712 is 38% above the average optimal value 21’400’252.

80% Perc.	σ[IBNR_i]
0	0
522	3'039'157
3'706	1'670'425
19'619	1'327'955
162'472	293'263
144'351	4'455'291
381'290	10'446'986
1'562'051	13'296'030
5'080'261	19'291'437
18'316'945	18'190'126
29'617'712	**31'979'525**

8 Conclusion

One cannot conclude without mentioning some further claims reserving topics that might suggest possible future developments.

Let us begin with two specific dependence modelling issues. First, claims reserving models assume independence between different accidents years. For this reason, they fail to model claims inflation appropriately, because claims inflation acts on all accident years simultaneously. A model that accounts for accident year dependence in runoff triangles has been proposed by Salzmann and Wüthrich [50]. Second, predictions of claims reserves often rely on individual loss triangles, where each triangle corresponds to a different line of business. Since different lines of business are often dependent it is necessary to develop models for loss triangle dependence. Examples that use copulas are Regis [48] and De Jong [8].

In view of the world-wide importance of solvency systems it is also necessary to adapt the classical claims reserving models. Some typical recent developments include Merz and Wüthrich [37], Hürlimann [22], Savelli and Clemente [51], Pirra et al. [41], Eling et al. [10], Salzmann [49] and Happ [14].

Among the many further methods, let us mention the development of claims reserving models based on multiple risk factors. Besides [52] and [22] we would like to point out [23], where the use of stochastic LDF’s is advocated.Finally, computational issues have rarely been taken into account. For example, to report IBNR reserves more frequently than the usual yearly periods, it is necessary to perform an extrapolation prior to the end of the first year and an interpolation for each successive development year. Early contributions include Sherman [56], and Robbin and Homer [47]. A more recent approach is Hürlimann [24].

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