Journal Homepage: -www.journalijar.com

INTERNATIONAL JOURNAL OF ADVANCED RESEARCH (IJAR)

Article DOI: 10.21474/IJAR01/18845
DOI URL: http://dx.doi.org/10.21474/IJAR01/18845

RESEARCH ARTICLE

FORMULATION OF A MODE I CRACK PROPAGATION MODEL BY ANALYZING THE STRESS FIELD AT THE CRACK TIP

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Manuscript Info

Manuscript History
Received: 05 April 2024
Final Accepted: 09 May 2024
Published: June 2024

Key words:-
Model, Propagation, Crack, Mode I, Constraints

Abstract

One of the phases in the development of materials is knowledge of their possible degradation. At this stage, taking into account the influence of temperature variation remains essential for certain materials. The objective of this work is to take this aspect into account by formulating a mode I crack propagation model by studying the stress field in the vicinity of the crack tip. To do this, starting from Hooke's law generalized in linear elasticity, we use the asymptotic expressions in displacements of the crack tip of Moes et Al. in order to obtain the stress field model. To validate the model, a comparison is made with data from the literature. The observations show a very high agreement of results. The advantage of this model is to take into account the role of temperature in the degradation of materials by analyzing crack propagation. It will be used in our future work.

Introduction:-

Composite materials, depending on their production conditions, are subject to differential shrinkage phenomena of the different constituents leading to residual cracks [1,2,3,4]. Under stress, we observe a propagation and intensification of these cracks depending on parameters such as temperature [5,6]

Our objective is to develop an analytical model of mode I crack propagation based on the analysis of the stress field at the crack tip taking into account the temperature variation. As a method, we first exploited the generalized Hooke's law and the small disturbance hypothesis. From these results, the asymptotic expressions in displacements of the crack tip [7,8], allowed us to obtain the desired model. A comparison is made with the results from the literature.

The observations made show that the established model gives a very good description of the crack propagation behavior in mode I.

Main working hypotheses

The material studied is a composite with two main phases: a polymer matrix and sufficiently short fiber reinforcements so that the dimensions of the composite are infinite. The specimen has an elastic, linear and isotropic behavior. The limits of use of the model to be established concern slow and sufficiently lower stresses in order to avoid sudden rupture of the material. The hypothesis of small disturbances will therefore be taken into account. In

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addition, if the composite undergoes a transformation, it must be continuous, infinitesimal and irreversible with an absence of chemical reactions and change of state.Figure 1. shows an illustration of the cracked specimen.

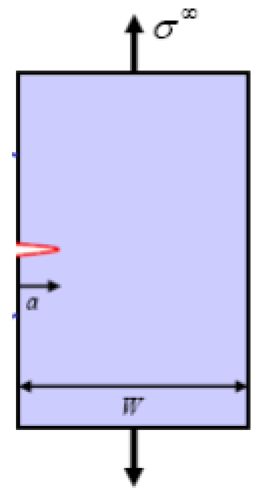

Fig. 1:- Illustration of the cracked composite [9].

Mode I stress field model at crack tip
Figure 2 illustrates the parameterization near the crack tip.

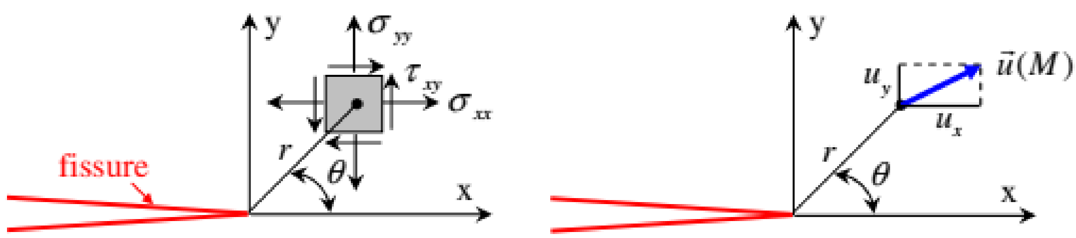

Fig. 2:- Parameterization a) in stresses and b) in displacement of the crack tip.

With the previous hypotheses, the generalized Hooke's law, in the absence of prestresses, is written:

$\bar{\bar{\sigma}} = 2\mu\bar{\bar{\varepsilon}} + [\lambda tr(\bar{\bar{\varepsilon}}) - \beta\Delta T]\bar{\bar{I}}$(1)

with : $\bar{\bar{\sigma}}$: tensor of the stress field at the crack tip (MPa)
$\bar{\bar{\varepsilon}}$: strain tensor near the crack front
$tr(\bar{\bar{\varepsilon}})$: trace of the strain tensor
$\bar{\bar{I}}$: tensor unit of order 4
μ : first Lamé coefficient or Coulomb modulus or transverse modulus of elasticity (MPa)
λ : second Lamé coefficient (MPa)
β : Thermal expansion coefficient of the composite ($^{o}C^{-1}$)
ΔT : variation of temperature from initial conditions (^{o}C)
Either in index form :

$\sigma_{ij} = 2\mu\varepsilon_{ij} + (\frac{2\mu\nu}{1-2\nu}\varepsilon_{kk} - \beta\Delta T)\delta_{ij} \quad , \ \forall i,j,k \in \{1;2\}$(2)

où : σ_{ij} : components of the stress tensor
ε_{ij} : components of the strain tensor
δ_{ij} : Kronecker symbol
ν : Poisson coefficient
In addition, the components of the linearized strain tensor under the hypothesis of small disturbances (HPP) are :

$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$(3)

with $u_{i,j}$ (respectively$u_{j,i}$) partial derivative of the displacement u_i (respectively u_j) according to x_j (respectively according to x_i).
From equations (2) and (3), we have :

$$\sigma_{ij} = 2\mu\left[\frac{1}{2}\left(u_{i,j} + u_{j,i}\right) + \frac{\nu}{1-2\nu}u_{k,k}\delta_{ij}\right] - \beta\Delta T\delta_{ij} \quad (4)$$

Then :

$$\begin{cases} \sigma_{xx} = 2\mu\left[u_{x,x} + \frac{\nu}{1-2\nu}\left(u_{x,x} + u_{y,y}\right)\right] - \beta\Delta T \\ \sigma_{yy} = 2\mu\left[u_{y,y} + \frac{\nu}{1-2\nu}\left(u_{y,y} + u_{x,x}\right)\right] - \beta\Delta T \\ \sigma_{xy} = \mu\left(u_{x,y} + u_{y,x}\right) \end{cases} \quad (5)$$

We will now calculate the components $u_{i,j}$ of the displacement in the vicinity of the crack tip.On a en mode I [7,8] :

$$\begin{cases} u_x = \frac{K_I}{2\mu}\sqrt{\frac{2r}{\pi}}\cos(\frac{\theta}{2})[1 - 2\nu + sin^2\left(\frac{\theta}{2}\right)] \\ u_y = \frac{K_I}{2\mu}\sqrt{\frac{2r}{\pi}}\sin(\frac{\theta}{2})[2(1-\nu) - cos^2\left(\frac{\theta}{2}\right)] \end{cases} \quad (6)$$

with : K_I : stress intensity factor (MPa$\sqrt{m}$).
Furthermore, we know that:

$$x = \mathrm{r}\cos(\theta)\ et y = rsin(\theta) \quad (7.1)$$

That’s to say :

$$r = \sqrt{x^2 + y^2} \text{ et } \theta = arccos(\frac{x}{\sqrt{x^2+y^2}}) \text{ (ou } \theta = arsin(\frac{x}{\sqrt{x^2+y^2}})\text{)} \quad (7.2)$$

x et y (respectively r et θ) being the coordinates in a Cartesian coordinate system (respectively cylindrical) of origin of the crack tip of a point very close to it.
From (6) and (7), we obtain :

$$\begin{cases} u_x = \frac{K_I}{2\mu}\sqrt{\frac{2}{\pi}}\cos(0{,}5\, arccos(\frac{x}{\sqrt{x^2+y^2}}))[1 - 2\nu + sin^2\left(0{,}5\, arccos(\frac{x}{\sqrt{x^2+y^2}})\right)](x^2 + y^2)^{1/4} \\ u_y = \frac{K_I}{2\mu}\sqrt{\frac{2}{\pi}}\sin(0{,}5\, arccos(\frac{x}{\sqrt{x^2+y^2}}))[2(1-\nu) - cos^2\left(0{,}5\, arccos(\frac{x}{\sqrt{x^2+y^2}})\right)](x^2 + y^2)^{1/4} \end{cases} \quad (8)$$

With, for an overflowing crack in relation to which the specimen is considered infinite [9], $K_I = 1{,}12\sigma^{\infty}\sqrt{\pi a}$ (MPa$\sqrt{m}$).(9)
Using relations 8, we therefore obtain the expressions of the partial derivatives u_(i,j) of the displacement field at the crack tip by setting:

$$u_{x,x} = \frac{\partial u_x}{\partial x}\ ;\ u_{x,y} = \frac{\partial u_x}{\partial y}\ ;\ u_{y,y} = \frac{\partial u_y}{\partial y}\ ;\ u_{y,x} = \frac{\partial u_y}{\partial x} \quad (10)$$

Let’s put : A = $\frac{K_I}{2\mu}\sqrt{\frac{2}{\pi}}$; B = $1 - 2\nu$ et C=$2(1-\nu)$
We demonstrate that (see Appendix I for details) :

$$u_{x,x} = A\frac{0,5y\,\sin(0,5\,arccos(\frac{x}{\sqrt{x^2+y^2}}))\left(B+sin^2\left(0,5\,arccos\left(\frac{x}{\sqrt{x^2+y^2}}\right)\right)\right)+cos\left(0,5\,arccos\left(\frac{x}{\sqrt{x^2+y^2}}\right)\right)\left(-ycos\left(0,5\,arccos\left(\frac{x}{\sqrt{x^2+y^2}}\right)\right)sin\left(0,5\,arccos\left(\frac{x}{\sqrt{x^2+y^2}}\right)\right)+0,5x\left(B+sin^2\left(0,5\,arccos\left(\frac{x}{\sqrt{x^2+y^2}}\right)\right)\right)\right)}{(x^2+y^2)^{\frac{3}{4}}}$$

$$u_{x,y} = A\frac{-0,5x\,sin(0,5\,arccos(\frac{x}{\sqrt{x^2+y^2}}))\left(B+sin^2\left(0,5\,arccos\left(\frac{x}{\sqrt{x^2+y^2}}\right)\right)\right)+cos\left(0,5\,arccos\left(\frac{x}{\sqrt{x^2+y^2}}\right)\right)\left(xcos\left(0,5\,arccos\left(\frac{x}{\sqrt{x^2+y^2}}\right)\right)sin\left(0,5\,arccos\left(\frac{x}{\sqrt{x^2+y^2}}\right)\right)+0,5y\left(B+sin^2\left(0,5\,arccos\left(\frac{x}{\sqrt{x^2+y^2}}\right)\right)\right)\right)}{(x^2+y^2)^{\frac{3}{4}}}$$

$$u_{y,y} = A\frac{0,5x\,cos(0,5\,arccos(\frac{x}{\sqrt{x^2+y^2}}))(C-cos^2(0,5\,\mathrm{arccos}\left(\frac{x}{\sqrt{x^2+y^2}}\right)))+sin\left(0,5\,arccos\left(\frac{x}{\sqrt{x^2+y^2}}\right)\right)\left(xsin\left(0,5\,arccos\left(\frac{x}{\sqrt{x^2+y^2}}\right)\right)cos\left(0,5\,arccos\left(\frac{x}{\sqrt{x^2+y^2}}\right)\right)+0,5y\left(C-cos^2\left(0,5\,arccos\left(\frac{x}{\sqrt{x^2+y^2}}\right)\right)\right)\right)}{(x^2+y^2)^{\frac{3}{4}}}$$

$$u_{y,x} = A\frac{-0,5y\,cos(0,5\,arccos(\frac{x}{\sqrt{x^2+y^2}}))(C-cos^2(0,5\,\mathrm{arccos}\left(\frac{x}{\sqrt{x^2+y^2}}\right)))+sin\left(0,5\,arccos\left(\frac{x}{\sqrt{x^2+y^2}}\right)\right)\left(-ysin\left(0,5\,arccos\left(\frac{x}{\sqrt{x^2+y^2}}\right)\right)cos\left(0,5\,arccos\left(\frac{x}{\sqrt{x^2+y^2}}\right)\right)+0,5x\left(C-cos^2\left(0,5\,arccos\left(\frac{x}{\sqrt{x^2+y^2}}\right)\right)\right)\right)}{(x^2+y^2)^{\frac{3}{4}}}$$

(11.1 ; 11.2 ; 11.3 ; 11.4)

Or according to cylindrical coordinates :

$$u_{x,x} = \frac{A}{2\sqrt{r}}\left[\cos(0{,}5\theta)\left(B + sin^2(0{,}5\theta)\right) - sin^2(\theta)\cos(0{,}5\theta)\right] \quad (12.1)$$

$$u_{x,y} = \frac{A}{2\sqrt{r}}\left[\sin(0{,}5\theta)\left(B + sin^2(0{,}5\theta)\right) + \sin(\theta)\cos(\theta)\cos(0{,}5\theta)\right] \quad (12.2)$$

$$u_{y,y} = \frac{A}{2\sqrt{r}}\left[\cos(0{,}5\theta)\left(C - cos^2(0{,}5\theta)\right) + sin(\theta)\, sin(0{,}5\theta)\cos(\theta)\right] \quad (12.3)$$

$$u_{y,x} = -\frac{A}{2\sqrt{r}}\left[\sin(0{,}5\theta)\left(C - cos^2(0{,}5\theta)\right) - sin^2(\theta)\sin(0{,}5\theta)\right] \quad (12.4)$$

Equations (5) combined with relations (12) constitute the stress field model sought in the vicinity of the crack tip (see Appendix II for details):

$$\begin{cases} \sigma_{xx} = \frac{K_I}{\sqrt{2\pi r}} f_{xx} - \beta\Delta T \\ \sigma_{yy} = \frac{K_I}{\sqrt{2\pi r}} f_{yy} - \beta\Delta T \\ \sigma_{xy} = \frac{K_I}{\sqrt{2\pi r}} f_{xy} - \beta\Delta T \end{cases} \quad (13)$$

Where the parameters f_{xx}, f_{yy} and f_{xy} are functions of space and behavior of the composite defined by:

$$\begin{cases} fxx = \frac{\sigma_{xx}}{\frac{K_I}{\sqrt{2\pi r}}} = [-\nu cos^3(0.5\theta) + 0.5\nu\, sin(2\theta)\, sin(0.5\theta) + cos(0.5\theta)\, sin^2(0.5\theta) + \\ \quad cos(0.5\theta)\, cos^2(\theta) - \nu\, cos(0.5\theta)\, sin^2(0.5\theta) - \nu\, cos(0.5\theta)\, cos^2(\theta)]/(1-2\nu) \\ fyy = \frac{\sigma_{yy}}{\frac{K_I}{\sqrt{2\pi r}}} = [2\, cos(0.5\theta) - cos^3(0.5\theta) + \nu cos^3(0.5\theta) + \nu\, cos(0.5\theta)\, sin^2(0.5\theta) \\ \quad +\nu\, cos(0.5\theta)\, cos^2(\theta) - 4\nu\, cos(0.5\theta) + 0.5\,(1-\nu) sin(2\theta)\, sin(0.5\theta)]/(1-2\nu) \\ fxy = \frac{\sigma_{xy}}{\frac{K_I}{\sqrt{2\pi r}}} = 0.5 sin(\theta)\, cos(1.5\theta) \end{cases} \quad (14)$$

It takes into account the size of the crack, the position of the point in the vicinity of the crack, the effect of temperature variation and the behavior of the composite.

For the validation of this model, we will carry out a comparative study with the results of [10].

Discussion and comparative study (validation)

The model of stress fields at the crack tip in the literature relating to the results of [10]is given by the following system :

$$\begin{cases} \sigma_{wxx} = \frac{K_I}{\sqrt{2\pi r}} fwxx \\ \sigma_{wyy} = \frac{K_I}{\sqrt{2\pi r}} fwyy \\ \sigma_{wxy} = \frac{K_I}{\sqrt{2\pi r}} fwxy \end{cases} \quad (15)$$

Where the parameters $fwxx$, $fwyy$ and $fwxy$ are space functions defined by :

$$\begin{cases} fwxx = \frac{\sigma_{wxx}}{\frac{K_I}{\sqrt{2\pi r}}} = \cos(\frac{\theta}{2})(1 - \sin(\frac{\theta}{2})\sin(\frac{3\theta}{2})) \\ fwyy = \frac{\sigma_{wyy}}{\frac{K_I}{\sqrt{2\pi r}}} = cos(\frac{\theta}{2})(1 + sin(\frac{\theta}{2})\, sin(\frac{3\theta}{2})) \\ fxwy = \frac{\sigma_{wxy}}{\frac{K_I}{\sqrt{2\pi r}}} = cos(\frac{\theta}{2})\, sin(\frac{\theta}{2})\cos(\frac{3\theta}{2}) \end{cases} \quad (16)$$

At constant temperature (condition imposed by Westergaard), the study of the validation of the proposed model amounts to comparing the parameters fxx and fwxx, fyy and fwyy, and fxy and fwxy.

For a better illustration, we will plot the graphs of the functions fwxx, fwyy and fwxy together with those of fxx, fyy and fxy respectively. For the latter, and for each axis, different graphs will be drawn according to the Poisson's ratio taken between 0.25 and 0.5; range of values for natural polymer and fiber materials [11]. The graphs are presented in Figures 3, 4 and 5.

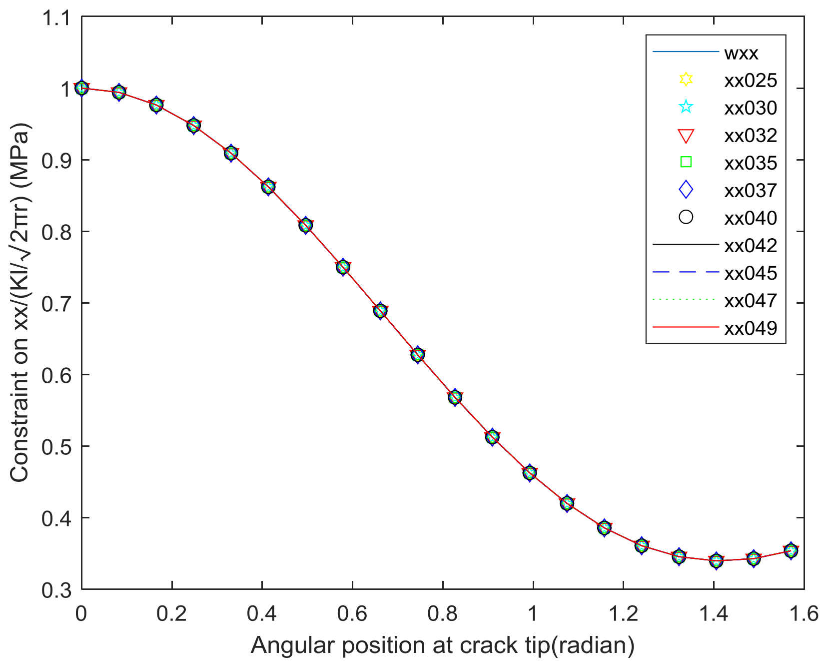

Figure 3:- Variation of the foncions f_{xx} et f_{wxx}

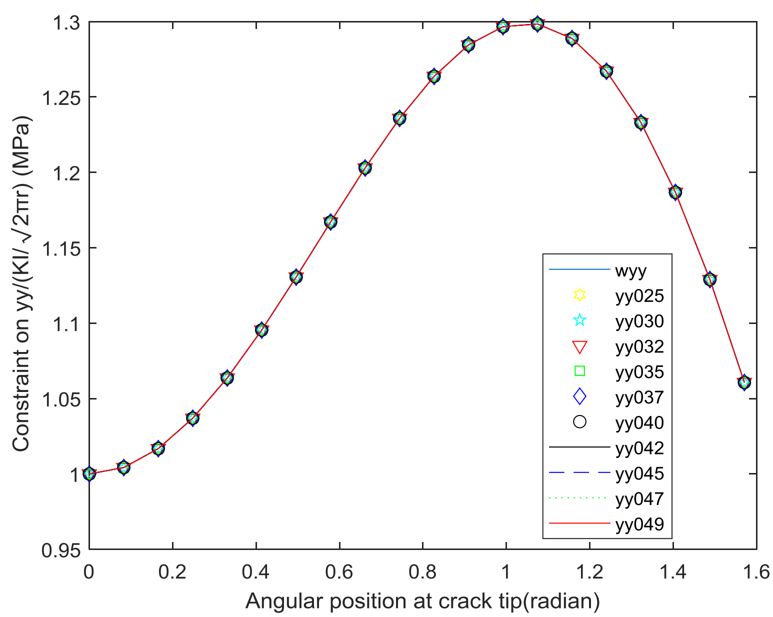

Figure 4:- Variation of the foncions f_{yy} et f_{wyy}

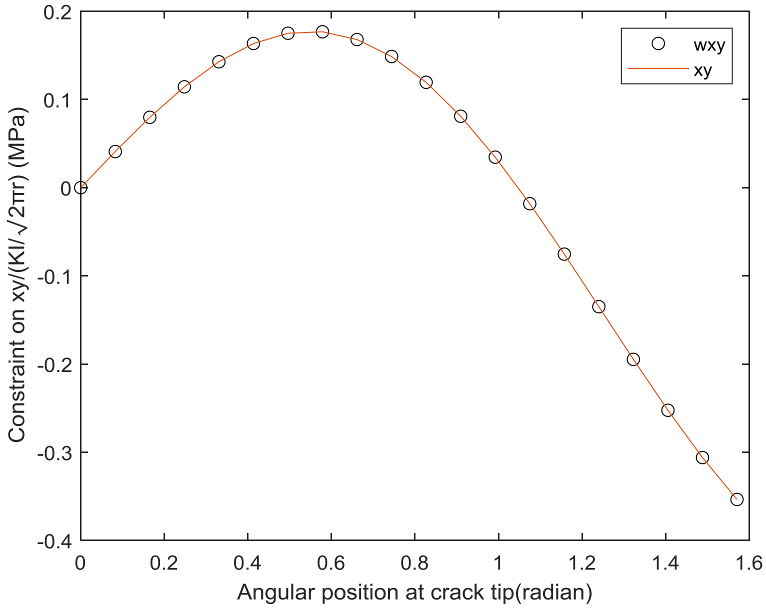

Figure 5:- Variation of the foncions f_{xy} et f_{wxy}

We observe that the graphs resulting from our model and those of [10] are identically superimposable. The margin for error is zero. This means that the measurements obtained are exactly equal to those in the literature. At constant temperature. However, in practice, this condition of no temperature variation is not realistic. Indeed, there is always a residual temperature variation which has an effect on the stress state of the material.

The interest of our developed model is to take into account this influence of temperature on the degradation of materials by analyzing crack propagation.

Conclusion:-

The development of new materials requires particular attention, especially for the prediction of their possible degradation. The analysis of crack propagation by studying fields at the crack tip is an effective means. Westergaard's asymptotic expressions are an example. In this work, it was a question of developing a model taking into account the effect of temperature. The results are compared to those in the literature. We have a very high agreement between the different measurements.

The interest of our developed model is to take into account this influence of temperature on the degradation of materials by analyzing crack propagation. This model will be used in some of our future work to study the degradation of materials that we will develop taking into account the influence of temperature.

Conflicts of Interest:

The authors declare that there are no conflicts of interest that could inappropriately influence, or be perceived to influence, the work reported in this manuscript.

Author's Contribution:

Ahmed Doumbia : writing manuscript, data collection, data analysis, and interpretation; Chardin Séri Séri and Seydou Traoré : data collection, reviewing and editing of the manuscript.

References:-

[1] Federica DAGHIA, Lionel GENDRE, David VIOLEAU, Les dégradations des matériaux composites : les phénomènes physiques, Université Paris-Saclay, ENS Cachan, France, 21/11/2011.

[2] Karen TRICONNET, Thèse, Identification Des Proprietes Mecaniques A Partir De Mesures De Champs Dans Un Materiau Multi-Phase, École Nationale Supérieure d'Arts et Métiers, Ecole doctorale n° 432 : Sciences des Métiers de l'Ingénieur, France, 25/06/2007.

[3] Corentin Musa, Élaboration et caractérisation de matériaux composites biosourcés à base de mucilage et de fibres de lin, Université du Littoral Côte d'Opale Unité de Chimie Environnementale et Interactions sur le vivant, UCEIV SFR Condorcet FR CNRS 3417, Dunkerque France, 4/11/2019.
[4] Abderrezak Gharbi, Analyse Des Fissurations Des Matériaux Composites Et Détermination De Leur Délaminage Utilisant Des Capteurs Piézo –Électriques, Universite Mentouri- Constantine Faculte Des Sciences De L'ingenieur Departement De Genie Mecanique, Algérie, 2005.
[5] Méjido HAJJAJ, Thèse, Propagation Dynamique Et Arret De Fissure De Clivage Dans Un Acier Bainitique, École Centrale Des Arts Et Manufactures « École Centrale Paris », 8/6/2006.
[6] [Irwin, 1957] Irwin, G.R. (1957), Analysis of stresses and strains near the end of a crack traversing a plate, Journal of Applied Mechanics, 24, 361-4
[7] Moes N, Dolbow J,Belytschko T. A finite element method for crack growth without remeshing. Int J Numer Methods Eng 1999. 46:131-150.
[8] Belytschko T, Black T. Elastic crack growth in finite element with minimal remeshing. International Journal of Numerical Methods in Engineering 1999. 45(5):601-620.
[9] Abderrahim Zeghloul, Cours, Concepts Fondamentaux De La Mecanique De La Rupture, Université de Loraine, 2006-2007
[10] Westergaard, H. M. Bearing Pressures and Cracks. Journal of Applied Mechanics 1939.Vol. 6:49-53.
[11] https://fr.wikipedia.org/wiki/Coefficient_de_Poisson (consulté le 19/08/2023)

Annex I : Calculation of partial derivatives of the displacement field at the crack tip

Partial derivatives $u_{i,j}$ of the displacement field at the crack tip by posing are calculated as follows:

$$u_{x,x}=\frac{\partial u_x}{\partial x}\ ;\ u_{x,y}=\frac{\partial u_x}{\partial y}\ ;\ u_{y,y}=\frac{\partial u_y}{\partial y}\ ;\ u_{y,x}=\frac{\partial u_y}{\partial x}$$

Calcuation of $u_{x,x}$

$$u_{x,x}=\frac{\partial}{\partial x}\left(\frac{K_I}{2\mu}\sqrt{\frac{2}{\pi}}cos(0,5\,cos^{-1}(\frac{x}{\sqrt{x^2+y^2}}))[1-2\nu+sin^2\left(0,5\,cos^{-1}(\frac{x}{\sqrt{x^2+y^2}})\right)](x^2+y^2)^{1/4}\right)$$

Y and b being taken here as constants, we apply the product rule: (fg)' = f'g+fg' with :

$$f=\frac{K_I}{2\mu}\sqrt{\frac{2}{\pi}}cos(0,5\,arccos(\frac{x}{\sqrt{x^2+y^2}})\ \text{ et }\ g=(1-2\nu+sin^2\left(0,5\,arccos\left(\frac{x}{\sqrt{x^2+y^2}}\right)\right))(x^2+y^2)^{1/4}$$

By puting A = $\frac{K_I}{2\mu}\sqrt{\frac{2}{\pi}}$ (we make a reservation of the multiplier constant A) et B = $1-2\nu$, we have :

$$u_{x,x}=A\frac{\partial}{\partial x}(\cos(0,5\,arccos(\frac{x}{\sqrt{x^2+y^2}})))(B+sin^2(0,5\text{arccos}(\frac{x}{\sqrt{x^2+y^2}})))\sqrt[4]{x^2+y^2}+A\frac{\partial}{\partial x}((B+sin^2(0,5arccos(\frac{x}{\sqrt{x^2+y^2}})))\sqrt[4]{x^2+y^2})\cos(0,5arccos(\frac{x}{\sqrt{x^2+y^2}}))$$

Moreover :

$$\frac{\partial}{\partial x}(\cos(0,5\,arccos(\frac{x}{\sqrt{x^2+y^2}})))=\frac{0,5y\sin(0,5\,arccos(\frac{x}{\sqrt{x^2+y^2}}))}{x^2+y^2}\quad \text{et}$$

$$\frac{\partial}{\partial x}\left(\left(B+sin^2\left(0,5arccos\left(\frac{x}{\sqrt{x^2+y^2}}\right)\right)\right)\sqrt[4]{x^2+y^2}\right)\cos\left(0,5arccos\left(\frac{x}{\sqrt{x^2+y^2}}\right)\right)$$

$$=\frac{-ycos\left(0,5\,\text{arccos}\left(\frac{x}{\sqrt{x^2+y^2}}\right)\right)\sin\left(0,5\,arccos\left(\frac{x}{\sqrt{x^2+y^2}}\right)\right)+0,5x\left(B+sin^2\left(0,5\,arccos\left(\frac{x}{\sqrt{x^2+y^2}}\right)\right)\right)}{(x^2+y^2)^{\frac{3}{4}}}$$

Then :

$$u_{x,x}=A\frac{0,5y\sin(0,5\,arccos(\frac{x}{\sqrt{x^2+y^2}}))}{x^2+y^2}\left(B+sin^2\left(0,5\,arccos\left(\frac{x}{\sqrt{x^2+y^2}}\right)\right)\right)\sqrt[4]{x^2+y^2}+A\frac{-ycos\left(0,5\,arccos\left(\frac{x}{\sqrt{x^2+y^2}}\right)\right)sin\left(0,5\,arccos\left(\frac{x}{\sqrt{x^2+y^2}}\right)\right)+0,5x\left(B+sin^2\left(0,5\,arccos\left(\frac{x}{\sqrt{x^2+y^2}}\right)\right)\right)}{(x^2+y^2)^{\frac{3}{4}}}cos\left(0,5arccos\left(\frac{x}{\sqrt{x^2+y^2}}\right)\right)$$

After simplification, we finally obtain:

$$u_{x,x}=A\frac{0,5y\,sin(0,5\,arccos(\frac{x}{\sqrt{x^2+y^2}}))\left(B+sin^2\left(0,5\,arccos\left(\frac{x}{\sqrt{x^2+y^2}}\right)\right)\right)+cos\left(0,5\,arccos\left(\frac{x}{\sqrt{x^2+y^2}}\right)\right)\left(-ycos\left(0,5\,arccos\left(\frac{x}{\sqrt{x^2+y^2}}\right)\right)sin\left(0,5\,arccos\left(\frac{x}{\sqrt{x^2+y^2}}\right)\right)+0,5x\left(B+sin^2\left(0,5\,arccos\left(\frac{x}{\sqrt{x^2+y^2}}\right)\right)\right)\right)}{(x^2+y^2)^{\frac{3}{4}}}$$

Or in cylindrical coordinates :

$$u_{x,x}=A\frac{0,5sin(\theta)\,sin(0,5\,\theta)\big(B+sin^2(0,5\theta)\big)+cos(0,5\theta)\big(-sin(\theta)cos(0,5\theta)\,sin(0,5\theta)+0,5cos(\theta)\big(B+sin^2(0,5\theta)\big)\big)}{\sqrt{r}}$$

After simplification :

$$u_{x,x}=\frac{A}{2\sqrt{r}}\left[\cos(0,5\theta)\big(B+sin^2\,(0,5\theta)\big)-sin^2(\theta)\cos(0,5\theta)\right]$$

Calculation of $u_{x,y}$

$$u_{x,y}=\frac{\partial}{\partial y}\left(\frac{K_I}{2\mu}\sqrt{\frac{2}{\pi}}cos(0,5\,cos^{-1}(\frac{x}{\sqrt{x^2+y^2}}))[1-2\nu+sin^2\left(0,5\,cos^{-1}(\frac{x}{\sqrt{x^2+y^2}})\right)](x^2+y^2)^{1/4}\right.$$

The same product rule allows us to write that :

$$u_{x,y} = A\frac{\partial}{\partial y}(\cos(0{,}5\,arccos(\frac{x}{\sqrt{x^2+y^2}})))(B+sin^2(0{,}5\arccos(\frac{x}{\sqrt{x^2+y^2}})))\sqrt[4]{x^2+y^2} + A\frac{\partial}{\partial y}((B + sin^2(0{,}5arccos(\frac{x}{\sqrt{x^2+y^2}})))\sqrt[4]{x^2+y^2})\cos(0{,}5arccos(\frac{x}{\sqrt{x^2+y^2}}))$$

with :

$$\frac{\partial}{\partial x}(\cos(0{,}5\,arccos(\frac{x}{\sqrt{x^2+y^2}}))) = -\frac{0{,}5x\sin(0{,}5\,arccos(\frac{x}{\sqrt{x^2+y^2}}))}{x^2+y^2}$$ et

$$\frac{\partial}{\partial x}\left(\left(B+sin^2\left(0{,}5arccos\left(\frac{x}{\sqrt{x^2+y^2}}\right)\right)\right)\sqrt[4]{x^2+y^2}\right)\cos\left(0{,}5arccos\left(\frac{x}{\sqrt{x^2+y^2}}\right)\right) = \frac{xcos\left(0{,}5\arccos\left(\frac{x}{\sqrt{x^2+y^2}}\right)\right)\sin\left(0{,}5\,arccos\left(\frac{x}{\sqrt{x^2+y^2}}\right)\right)+0{,}5y\left(B+sin^2\left(0{,}5\,arccos\left(\frac{x}{\sqrt{x^2+y^2}}\right)\right)\right)}{(x^2+y^2)^{\frac{3}{4}}}$$

Then :

$$u_{x,y} = -A\frac{0{,}5x\sin(0{,}5\,arccos(\frac{x}{\sqrt{x^2+y^2}}))}{x^2+y^2}\left(B+sin^2\left(0{,}5\,arccos\left(\frac{x}{\sqrt{x^2+y^2}}\right)\right)\right)\sqrt[4]{x^2+y^2} + A\frac{xcos\left(0{,}5\,arccos\left(\frac{x}{\sqrt{x^2+y^2}}\right)\right)sin\left(0{,}5\,arccos\left(\frac{x}{\sqrt{x^2+y^2}}\right)\right)+0{,}5y\left(B+sin^2\left(0{,}5\,arccos\left(\frac{x}{\sqrt{x^2+y^2}}\right)\right)\right)}{(x^2+y^2)^{\frac{3}{4}}}$$

Either in cylindrical coordinates:

$$u_{x,y} = A\frac{-0{,}5cos(\theta)\,sin(0{,}5\,\theta)\big(B+sin^2(0{,}5\theta)\big)+cos(0{,}5\theta)(cos(\theta)cos(0{,}5\theta)\,sin(0{,}5\theta)+0{,}5sin(\theta)\big(B+sin^2(0{,}5\theta)\big))}{\sqrt{r}}$$

After simplification :

$$u_{x,y} = \frac{A}{2\sqrt{r}}[\sin(0{,}5\theta)\big(B+sin^2(0{,}5\theta)\big)+\sin(\theta)\cos(\theta)\cos(0{,}5\theta)]$$

Calculation of $\boldsymbol{u_{y,y}}$

$$u_{y,y} = \frac{\partial}{\partial y}\left(\frac{K_I}{2\mu}\sqrt{\frac{2}{\pi}}si\,n\left(0{,}5\,arccos\left(\frac{x}{\sqrt{x^2+y^2}}\right)\right)\left[2(1-\nu)-cos^2\left(0{,}5\,arccos\left(\frac{x}{\sqrt{x^2+y^2}}\right)\right)\right](x^2+y^2)^{1/4}\right)$$

By puting C=$2(1-\nu)$, it comes after application of the product rule :

$$u_{y,y} = A\frac{\partial}{\partial y}\left(\sin\left(0{,}5\arccos\left(\frac{x}{\sqrt{x^2+y^2}}\right)\right)\right)(C-cos^2\left(0{,}5\,arccos\left(\frac{x}{\sqrt{x^2+y^2}}\right)\right))\sqrt[4]{x^2+y^2} + A\frac{\partial}{\partial y}(\left(Ccos^2\left(0{,}5\,arccos\left(\frac{x}{\sqrt{x^2+y^2}}\right)\right)\right)\sqrt[4]{x^2+y^2})\sin(0{,}5\,arccos\left(\frac{x}{\sqrt{x^2+y^2}}\right))$$

We have :

$$\frac{\partial}{\partial y}\left(\sin\left(0{,}5\arccos\left(\frac{x}{\sqrt{x^2+y^2}}\right)\right)\right) = \frac{0{,}5x\,cos(0{,}5\,arccos(\frac{x}{\sqrt{x^2+y^2}}))}{x^2+y^2}$$ and

$$\frac{\partial}{\partial y}\left(\left(C-cos^2\left(0{,}5\,arccos\left(\frac{x}{\sqrt{x^2+y^2}}\right)\right)\right)\sqrt[4]{x^2+y^2}\right) = \frac{xsin\left(0{,}5\,arccos\left(\frac{x}{\sqrt{x^2+y^2}}\right)\right)cos\left(0{,}5\,arccos\left(\frac{x}{\sqrt{x^2+y^2}}\right)\right)+0{,}5y\left(C-cos^2\left(0{,}5\,arccos\left(\frac{x}{\sqrt{x^2+y^2}}\right)\right)\right)}{(x^2+y^2)^{\frac{3}{4}}}$$

Then:

$$u_{y,y} = A\frac{0{,}5x\cos(0{,}5\,arccos(\frac{x}{\sqrt{x^2+y^2}}))}{x^2+y^2}\left(C-cos^2\left(0{,}5\,arccos\left(\frac{x}{\sqrt{x^2+y^2}}\right)\right)\right)\sqrt[4]{x^2+y^2}$$
$$+\frac{Axsin\left(0{,}5\,arccos\left(\frac{x}{\sqrt{x^2+y^2}}\right)\right)cos\left(0{,}5\,arccos\left(\frac{x}{\sqrt{x^2+y^2}}\right)\right)+0{,}5y\left(C-cos^2\left(0{,}5\,arccos\left(\frac{x}{\sqrt{x^2+y^2}}\right)\right)\right)}{(x^2+y^2)^{\frac{3}{4}}}sin(0{,}5\,arccos\left(\frac{x}{\sqrt{x^2+y^2}}\right))$$

Either in cylindrical coordinates:

$$u_{y,y} = A\frac{0{,}5\cos(\theta)\,cos(0{,}5\,\theta)(C-cos^2(0{,}5\theta))+sin(0{,}5\theta)\,(\cos(\theta)sin(0{,}5\theta)\,cos(0{,}5\theta)+0{,}5\sin(\theta)\left(C-cos^2(0{,}5\theta)\right))}{\sqrt{r}}$$

After simplification :

$$u_{y,y} = \frac{A}{2\sqrt{r}}\left[\cos(0{,}5\theta)(C-cos^2\,(0{,}5\theta))+sin(\theta)\,sin(0{,}5\theta)\cos(\theta)\right]$$

Calcul de $u_{y,x}$

$$u_{y,x} = \frac{\partial}{\partial y}\left(\frac{K_I}{2\mu}\sqrt{\frac{2}{\pi}}si\,n\left(0{,}5\,arccos\left(\frac{x}{\sqrt{x^2+y^2}}\right)\right)\left[C-cos^2\left(0{,}5\,arccos\left(\frac{x}{\sqrt{x^2+y^2}}\right)\right)\right](x^2+y^2)^{1/4}\right)$$
$$u_{y,x} = A\frac{\partial}{\partial x}\left(\sin\left(0{,}5\,\mathrm{arccos}\left(\frac{x}{\sqrt{x^2+y^2}}\right)\right)\right)(C-cos^2\left(0{,}5\,arccos\left(\frac{x}{\sqrt{x^2+y^2}}\right)\right))\sqrt[4]{x^2+y^2}$$
$$+A\frac{\partial}{\partial x}(\left(C-cos^2\left(0{,}5\,arccos\left(\frac{x}{\sqrt{x^2+y^2}}\right)\right)\right)\sqrt[4]{x^2+y^2})\,\sin(0{,}5\,arccos\left(\frac{x}{\sqrt{x^2+y^2}}\right))$$

We have :

$$\frac{\partial}{\partial x}\left(\sin\left(0{,}5\,\mathrm{arccos}\left(\frac{x}{\sqrt{x^2+y^2}}\right)\right)\right) = \frac{-0{,}5y\,cos\,(\,0{,}5\,arccos\,(\frac{x}{\sqrt{x^2+y^2}}))}{x^2+y^2}$$ et

$$\frac{\partial}{\partial x}\left(\left(C-cos^2\left(0{,}5\,arccos\left(\frac{x}{\sqrt{x^2+y^2}}\right)\right)\right)\sqrt[4]{x^2+y^2}\right) = \frac{-ysin\left(0{,}5\,arccos\left(\frac{x}{\sqrt{x^2+y^2}}\right)\right)cos\left(0{,}5\,arccos\left(\frac{x}{\sqrt{x^2+y^2}}\right)\right)+0{,}5x\left(C-cos^2\left(0{,}5\,arccos\left(\frac{x}{\sqrt{x^2+y^2}}\right)\right)\right)}{(x^2+y^2)^{\frac{3}{4}}}:$$

$$u_{y,x} = -A\frac{0{,}5y\cos(0{,}5\,arccos(\frac{x}{\sqrt{x^2+y^2}}))}{x^2+y^2}\left(C-cos^2\left(0{,}5\,arccos\left(\frac{x}{\sqrt{x^2+y^2}}\right)\right)\right)\sqrt[4]{x^2+y^2}$$
$$+\frac{Aysin\left(0{,}5\,arccos\left(\frac{x}{\sqrt{x^2+y^2}}\right)\right)cos\left(0{,}5\,arccos\left(\frac{x}{\sqrt{x^2+y^2}}\right)\right)+0{,}5x\left(C-cos^2\left(0{,}5\,arccos\left(\frac{x}{\sqrt{x^2+y^2}}\right)\right)\right)}{(x^2+y^2)^{\frac{3}{4}}}sin(0{,}5\,arccos\left(\frac{x}{\sqrt{x^2+y^2}}\right))$$

Either in cylindrical coordinates :

$$u_{y,x} = A\frac{-0{,}5\sin(\theta)\,cos(0{,}5\,\theta)(C-cos^2(0{,}5\theta))+sin(0{,}5\theta)\,(-\sin(\theta)sin(0{,}5\theta)\,cos(0{,}5\theta)+0{,}5\cos(\theta)\left(C-cos^2(0{,}5\theta)\right))}{\sqrt{r}}$$

After simplification :

$$u_{y,x} = -\frac{A}{2\sqrt{r}}[\sin(0{,}5\theta)\left(C-cos^2(0{,}5\theta)\right)-sin^2(\theta)\sin(0{,}5\theta)]$$

<u>Annex II</u> : Calculation of the stress field at the crack tip

Calcul de σ_{xx}

$$\sigma_{xx} = 2\mu\left[u_{x,x} + \frac{\nu}{1-2\nu}\left(u_{x,x} + u_{y,y}\right)\right] - \beta\Delta T$$

Taking into account the partial derivatives of the displacement in appendix I and replacing A, B and C by their expressions, we have:

$$\sigma_{xx} = \frac{K_I}{(1-2\nu)\sqrt{2\pi r}}[(1-\nu)\cos(0.5\theta)\left(1-2\nu+sin^2(0.5\theta)-sin^2(\theta)\right)+\nu[\cos(0.5\theta)\left(2-2\nu-cos^2(0.5\theta)\right) + 0.5\sin(2\theta)\sin(0.5\theta)]] - \beta\Delta T$$

$$\sigma_{xx} = \frac{K_I}{(1-2\nu)\sqrt{2\pi r}}\left[\nu\left(\cos(0.5\theta)\left(2-2\nu-cos^2(0.5\theta)\right)+0.5\sin(2\theta)\sin(0.5\theta)\right) + (1-\nu)\cos(0.5\theta)\left(sin^2(0.5\theta)+cos^2(\theta)-2\nu\right)\right] - \beta\Delta T$$

$$\sigma_{xx} = \frac{K_I}{(1-2\nu)\sqrt{2\pi r}}[\nu(2\cos(0.5\theta)-2\nu\cos(0.5\theta)-cos^3(0.5\theta)+0.5\sin(2\theta)\sin(0.5\theta)) + (1-\nu)\cos(0.5\theta)\left(sin^2(0.5\theta)+cos^2(\theta)-2\nu\right)] - \beta\Delta T$$

$$\sigma_{xx} = \frac{K_I}{(1-2\nu)\sqrt{2\pi r}}[2\,\nu cos(0.5\theta)-2\nu^2\,cos(0.5\theta)-\nu cos^3(0.5\theta)+0.5\,sin(2\theta)\,sin(0.5\theta) + (1-\nu)\,cos(0.5\theta)\left(sin^2(0.5\theta)+cos^2(\theta)-2\nu\right)] - \beta\Delta T$$

$$\sigma_{xx} = \frac{K_I}{(1-2\nu)\sqrt{2\pi r}}[2\,\nu cos(0.5\theta)-2\nu^2\,cos(0.5\theta)-\nu cos^3(0.5\theta)+0.5\,sin(2\theta)\,sin(0.5\theta)-\nu\cos(0.5\theta)\,sin^2(0.5\theta) - \nu\cos(0.5\theta)\,cos^2(\theta)+2\nu^2\cos(0.5\theta)+cos(0.5\theta)sin^2(0.5\theta)+\cos(0.5)cos^2(\theta)-2\nu\cos(0.5\theta)] - \beta\Delta T$$

After grouping terms by terms and simplification, we obtain:

$$\sigma_{xx} = \frac{K_I}{(1-2\nu)\sqrt{2\pi r}}[-\nu cos^3(0.5\theta)+0.5\nu\sin(2\theta)\sin(0.5\theta)+\cos(0.5\theta)\,sin^2(0.5\theta)+\cos(0.5\theta)\,cos^2(\theta) - \nu\,cos(0.5\theta)\,sin^2(0.5\theta)-\nu\,cos(0.5\theta)\,cos^2(\theta)] - \beta\Delta T$$

Calculation of $\boldsymbol{\sigma_{yy}}$

$$\sigma_{yy} = 2\mu\left[u_{y,y} + \frac{\nu}{1-2\nu}\left(u_{y,y} + u_{x,x}\right)\right] - \beta\Delta T$$

$$\sigma_{yy} = \frac{K_I}{(1-2\nu)\sqrt{2\pi r}}[\nu\cos(0.5\theta)\left(1-2\nu+sin^2(0.5\theta)-sin^2(\theta)\right)+(1-\nu)[\cos(0.5\theta)\left(2-2\nu-cos^2(0.5\theta)\right) + 0.5\sin(2\theta)\sin(0.5\theta)]] - \beta\Delta T$$

$$\sigma_{yy} = \frac{K_I}{(1-2\nu)\sqrt{2\pi r}}\left[\nu\cos(0.5\theta)\left(sin^2(0.5\theta)+cos^2(\theta)-2\nu\right) + (1-\nu)\left(\cos(0.5\theta)\left(2-2\nu-cos^2(0.5\theta)\right)+0.5\sin(2\theta)\sin(0.5\theta)\right)\right] - \beta\Delta T$$

$$\sigma_{yy} = \frac{K_I}{(1-2\nu)\sqrt{2\pi r}}[\nu\cos(0.5\theta)\left(sin^2(0.5\theta)+cos^2(\theta)-2\nu\right) + (1-\nu)(2\,cos(0.5\theta)-2\,\nu cos(0.5\theta)-cos^3(0.5\theta)+0.5\sin(2\theta)\sin(0.5\theta))] - \beta\Delta T$$

$$\sigma_{yy} = \frac{K_I}{(1-2\nu)\sqrt{2\pi r}}[\nu\cos(0.5\theta)\,sin^2(0.5\theta)+\nu\nu\,cos(0.5\theta)\,cos^2(\theta)-2\nu^2\cos(0.5\theta) + (1-\nu)(2\,cos(0.5\theta)-2\nu\,cos(0.5\theta)-cos^3(0.5\theta)+0.5sin(2\theta)\,si\,n(0.5\theta))] - \beta\Delta T$$

After grouping terms by terms and simplification, we obtain:

$$\sigma_{yy} = \frac{K_I}{(1-2\nu)\sqrt{2\pi r}}[2\,cos(0.5\theta)-cos^3(0.5\theta)+\nu cos^3(0.5\theta)+\nu\cos(0.5\theta)\,sin^2(0.5\theta)+\nu\,cos(0.5\theta)\,cos^2(\theta)-4\nu\cos(0.5\theta) + 0.5\,(1-\nu)\sin(2\theta)\sin(0.5\theta)] - \beta\Delta T$$

Calculation of $\boldsymbol{\sigma_{xy}}$

$$\sigma_{xy} = \mu\left(u_{x,y} + u_{y,x}\right)$$

$$\sigma_{yy} = \frac{K_I}{2\sqrt{2\pi r}}\left[sin(0.5\theta)\left(B+sin^2(0.5\theta)\right)+sin(\theta)\,cos(\theta)\,cos(0.5\theta)]+sin(0.5\theta)\left(C-cos^2(0.5\theta)\right)-sin^2(\theta)\,sin(0.5\theta)\right] - \beta\Delta T$$

After simplification, we have :

$$\sigma_{xy} = \frac{K_I}{2\sqrt{2\pi r}}\sin(\theta)\cos(1.5\theta)$$