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RESEARCH ARTICLE

Dirichlet Averages of Generalized Fox-Wright Function

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Abstract

The objective of this paper is to investigate the Dirichlet averages of the generalized Fox-Wright hypergeometric function introduced by Wright in (1935)[9,10],Like the functions of the Mittag-Leffler type, the functions of the Wright type are known to play fundamental roles in various applications of the fractional calculus. This is mainly due to the fact that they are interrelated with the Mittag-Leffler functions through Laplace and Fourier transforms.
The authors making the use of Riemann – Liouville integrals andDirichletintegrals which is a multivariate integral and the generalization of a beta integral.
Finally, the authors deduce representations for the Dirichlet averages $R_k(\beta,\beta^{|};x,y)$ of the generalized Fox-Wright function with the fractional integrals in particular Riemann – Liouville integrals.Special cases of the established results associated with generalized Fox-Wright functions have been discussed.

Introduction

The Dirichletaverages of a function are a certain kind integral average with respect to Dirichlet measure. The concept of Dirichlet average was introduced by Carlson in 1977. It is studied among others by Carlson [1,2,4], Zu Castell [5], Massopust and Forster[6], Neuman and Vanfleet[7] and others . A detailed and comprehensive account of various types of Dirichlet averages hasbeen given by Carlson in his monographs [3].In this paper Dirichlet averages of the generalized Fox-Wright due to Wright[9,10] have been studied by the authors.
This paper is devoted to investigation of the generalized Fox- Wright functions (also known as Fox-Wright psi function or just Wright function) is a generalization of the generalized hypergeometric function ${}_pF_q(z)$ based on an idea of E. Maitland Wright (1935).

$$ {}_p\psi_q\left(\begin{matrix}(a_1;A_1)(a_2;A_2) & \cdots & (a_p;A_p) \\ & & \vdots \\ (b_1;B_1)(b_2;B_2) & \cdots & (b_q;B_q)\end{matrix}\middle| z\right) = \sum_{n=0}^{\infty}\frac{\Gamma(a_1+nA_1)\Gamma(a_2+nA_2)\ldots\Gamma(a_p+nA_p)}{\Gamma(b_1+nB_1)\Gamma(b_2+nB_2)\ldots\Gamma(b_q+nB_q)}\frac{z^n}{n!} \qquad (1) $$

Where $1+\sum_{j=1}^{q}B_j - \sum_{j=1}^{p}A_j \geq 0$(equality only holds for appropriately bounded z. The Fox-Wright function is a special case of the Fox- H-function (Srivastava 1984)[12].

$${}_p\Psi_q\left(\begin{matrix}(a_1;A_1)(a_2;A_2) & \cdots & (a_p;A_p)\\ & & \vdots\\ (b_1;B_1)(b_2;B_2) & \cdots & (b_q;B_q)\end{matrix}\middle| z\right) =$$

$$H_{p,q+1}^{1,p}\left(\begin{matrix}(1-a_1;A_1)(1-a_2;A_2) & \cdots & (1-a_p;A_p)\\ & & \vdots\\ (0,1)(1-b_1;B_1)(1-b_2;B_2) & \cdots & (1-b_q;B_q)\end{matrix}\middle| -z\right) \quad (2)$$

It follows from (2) that generalized Mittag- Leffler function $E_{\alpha,\beta}^{\gamma}(z)$ can be represented in terms of the Wright function as

$$E_{\alpha,\beta}^{\gamma}(z) = \frac{1}{\Gamma(\gamma)}{}_1\Psi_1\left(\begin{matrix}(\gamma;1) & \cdots & \\ & & \vdots\\ (\beta;\alpha) & \cdots & \end{matrix}\middle| z\right)$$

$$E_{\alpha,\beta}(z) = {}_1\Psi_1\left(\begin{matrix}(1;1) & \cdots & \\ & & \vdots\\ (\beta;\alpha) & \cdots & \end{matrix}\middle| z\right) = H_{1,2}^{1,1}\left(\begin{matrix}(1-a_1;A_1) & \cdots & \\ & & \vdots\\ (0,1)(1-b_1;B_1) & \cdots & \end{matrix}\middle| -z\right)$$

Definitions: We give below some definitions which are necessary in this paper.
Standard simplex in $R^n, n \geq 1$: Wedenote the standard simplex in $R^n, n \geq 1$ by

E = E_n = $(u_1 \;, u_2, \dots u_n); u_1 \geq 0, u_2 \geq 0, \dots, u_n \geq 0$ and $u_1 + u_2 + u_3 + u_n \leq 1\}$.

Dirichlet Measures: let b ϵc^k> ; K ≥ 2 and let E = E_{k-1} be the standard simplex in R^{k-1} $.The$ complex measure μ_b defined by [1]

$$d_{\mu_b}(\mathrm{u}) = \frac{1}{B(b)} u_1^{\;b_1-1} u_2^{\;b_2-1} u_3^{\;b_3-1} \dots u_k^{\;b_{k-1}-1}(1-u_1 \;, 1-u_2, \dots 1-u_{k-1})^{b_k-1} d_{u_1} d_{u_2} d_{u_3} \dots d_{u_{k-1}}.$$

Here B(b) = B$(b_1 \;, b, \dots b_k) = \frac{\Gamma(b_1)\Gamma(b_2)\dots\Gamma(b_k)}{\Gamma(b_1+b_2+\cdots . b_k)}$

C> = {zϵc : z≠0 }

Dirichlet average: let Ωbe aconvex set in C and let z= $(z_1 \;, z_2, \dots z_n)$ $\epsilon\Omega^n$, n≥2, and let f be a measurable function on Ω .Define

F(b; z) = $\int_{E_{n-1}} f(uoz)\, d_{\mu_b}$(u).where d_{μ_b}(u) is a Dirichlet Measure.

B(b) = B$(b_1 \;, b, \dots b_n) = \frac{\Gamma(b_1)\Gamma(b_2)\dots\Gamma(b_n)}{\Gamma(b_1+b_2+\cdots . b_n)}$, R$(b_j)$ > 0, j = 1,2,3…,n,

and $uoz = \sum_{j=1}^{n-1} u_j z_j + (1-u_1 \;\dots - u_{n-1})\, z_n$.

For n= 1, $f(b; z)$ = f(z) , for n = 2 , we have

$d_{\mu_{\beta,\beta'}}(u) = \frac{\Gamma(\beta+\beta')}{\Gamma(\beta)\Gamma(\beta')} u^{\beta-1}(1-u)^{\beta'-1} d(u).$

Carlson [3] investigated the average for
$f(z) = z^k$, kϵR,
$R_k(b;z) = \int_{E_{n-1}} (uoz)^k \, d_{\mu_b}(u)$, (kϵR).

And for n= 2 , Carlson proved that []

$$R_k(\beta,\beta';x,y) = \frac{1}{B(\beta,\beta')}\int_0^1 [ux+(1-u)y]^k u^{\beta-1}(1-u)^{\beta'-1} d(u),$$

Where $\beta,\beta' \epsilon C$, min [R(β),R(β')] > 0 , $x, y\epsilon R$.
Our paper is devoted to the study of theDirichlet averages of the generalized Fox- Wright function (1) in the form

$$\text{pMq}\left[\left(\begin{matrix}(a_1;A_1)(a_2;A_2) & \cdots & (a_p;A_p)\\ & & \vdots \\ (b_1;B_1)(b_2;B_2) & \cdots & (b_q;B_q)\end{matrix}\middle|(\beta,\beta';x,y)\right)\right]=\int_{E_1} \text{p}\psi\text{q}(uoz)\, d_{\mu_{\beta\beta'}}(u) \qquad (3).$$

Where R(β) > 0, R(β') > 0 ; $x, y\epsilon R$ and $\beta,\beta'\epsilon$ C.
Reimann-Liouville fractional integral of order αϵ C, R(α)> 0 [13].

$$(I_{a+}^{\alpha} f)x = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1} f(t)\, dt , (x > a , a\epsilon R). \qquad (4)$$

Representation of R_k and pMq in terms of Reimann-Liouville Fractional Integrals

In this section we deduced representations for the Dirichlet averages $R_k(\beta,\beta',x,y)$ andpMq$(\beta,\beta';x,y)$with fractional integral operators.

Theorem : Let$\beta,\beta'\epsilon$ Complex numbers , R(β) > 0, R(β') > 0 , and x,y be real numbers such that $x > y$and $1+\sum_{j=1}^{q} B_j - \sum_{j=1}^{p} A_j \geq 0$,and pMqand I_{a+}^{α} be given by (3) and (4) respectively . Then the Dirichlet average of the generalized Fox- Wright functions is given by

$$\text{pMq}\left[\left(\begin{matrix}(a_1;A_1)(a_2;A_2) & \cdots & (a_p;A_p)\\ & & \vdots \\ (b_1;B_1)(b_2;B_2) & \cdots & (b_q;B_q)\end{matrix}\middle|(\beta,\beta';x,y)\right)\right]$$
$$= \frac{\Gamma(\beta+\beta')}{\Gamma(\beta)(x-y)^{\beta+\beta'-1}}\left[(I_{0+}^{\alpha}\, \text{p}\psi\text{q}\left(\begin{matrix}(a_1;A_1)(a_2;A_2) & \cdots & (a_p;A_p)\\ & & \vdots \\ (b_1;B_1)(b_2;B_2) & \cdots & (b_q;B_q)\end{matrix}\middle| z\right)\right].$$

Where $\beta,\beta'\epsilon$C , R(β) > 0, R(β') > 0 , x,y ϵ R and $1+\sum_{j=1}^{q} B_j - \sum_{j=1}^{q} A_j \geq 0$ (equality only holds for appropriately bounded z).

Proof : According to equation (1) and (2) we have,

$$\text{pMq}\left[\left(\begin{matrix}(a_1;A_1)(a_2;A_2) & \cdots & (a_p;A_p)\\ & & \vdots \\ (b_1;B_1)(b_2;B_2) & \cdots & (b_q;B_q)\end{matrix}\middle|(\beta,\beta^{|};\text{x},\text{y})\right)\right] =$$

$$\frac{1}{B(\beta,\beta^{|})}\sum_{n=0}^{\infty}\frac{\prod_{j=1}^{p}\Gamma(a_j+nA_j)}{\prod_{j=1}^{q}\Gamma(b_j+nB_j)}\frac{1}{n!}\int_0^1[y+u(x-y)]^n\,u^{\beta-1}(1-u)^{\beta^{|}-1}\text{d(u)}.$$

$$\text{pMq}\left[\left(\begin{matrix}(a_1;A_1)(a_2;A_2) & \cdots & (a_p;A_p)\\ & & \vdots \\ (b_1;B_1)(b_2;B_2) & \cdots & (b_q;B_q)\end{matrix}\middle|(\beta,\beta^{|};\text{x},\text{y})\right)\right]$$

$$=\frac{\Gamma(\beta+\beta^{|})}{\Gamma(\beta^{|})\Gamma(\beta)}\sum_{n=0}^{\infty}\frac{\prod_{j=1}^{p}\Gamma(a_j+nA_j)}{\prod_{j=1}^{q}\Gamma(b_j+nB_j)}\frac{1}{n!}\int_0^1[y+u(x-y)]^n\,u^{\beta-1}(1-u)^{\beta^{|}-1}\text{d(u)}.$$

Put $u(x-y)=t$ in above equation , we get

$$\text{pMq}\left[\left(\begin{matrix}(a_1;A_1)(a_2;A_2) & \cdots & (a_p;A_p)\\ & & \vdots \\ (b_1;B_1)(b_2;B_2) & \cdots & (b_q;B_q)\end{matrix}\middle|(\beta,\beta^{|};\text{x},\text{y})\right)\right]$$

$$=\frac{\Gamma(\beta+\beta^{|})}{\Gamma(\beta^{|})\Gamma(\beta)}\sum_{n=0}^{\infty}\frac{\prod_{j=1}^{p}\Gamma(a_j+nA_j)}{\prod_{j=1}^{q}\Gamma(b_j+nB_j)}\frac{1}{n!}\int_0^{x-y}[y+t]^n\left\{\frac{t}{x-y}\right\}^{\beta-1}\left(1-\frac{t}{x-y}\right)^{\beta^{|}-1}\frac{dt}{x-y}$$

$$=\frac{(x-y)^{1-\beta-\beta^{|}}}{B(\beta,\beta^{|})}\sum_{n=0}^{\infty}\frac{\prod_{j=1}^{p}\Gamma(a_j+nA_j)}{\prod_{j=1}^{q}\Gamma(b_j+nB_j)}\frac{1}{n!}\int_0^{x-y}[y+t]^n\,\{t\}^{\beta-1}(x-y-1)^{\beta^{|}-1}dt$$

$$=\frac{(x-y)^{1-\beta-\beta^{|}}}{B(\beta,\beta^{|})}\int_0^{x-y} t^{\beta-1}\,\text{p}\psi\text{q}\left(\begin{matrix}(a_j;A_j)_{1,p}\\(b_j;B_j)_{1,q}\end{matrix}\middle|y+t\right)(x-y-1)^{\beta^{|}-1}dt$$

$$=\frac{(x-y)^{1-\beta-\beta^{|}}}{B(\beta,\beta^{|})}\int_0^{x-y} t^{\beta-1}\,\text{p}\psi\text{q}\left(\begin{matrix}(a_j;A_j)_{1,p}\\(b_j;B_j)_{1,q}\end{matrix}\middle|y+t\right)(x-y-1)^{\beta^{|}-1}dt.$$

This proves the theorem.

Special Cases

In this section, we consider some particular cases of the above theorem by setting p=q=1 and $a=\gamma$, A = 1, b= β and B= α, we get well known result reported in [8] as follows

$$M_{\alpha,\delta}^{\gamma}(\beta,\beta^{|};\text{x},\text{y})=\frac{\Gamma(\beta+\beta^{|})}{\Gamma(\beta)(x-y)^{\beta+\beta^{|}-1}}\int_0^{x-y}t^{\beta-1}E_{\alpha,\beta}^{\gamma}(y+t)\,(x-y-1)^{\beta^{|}-1}dt.$$

$$M_{\alpha,\delta}^{\gamma}(\beta,\beta^{|};\text{x},\text{y})=\frac{\Gamma(\beta+\beta^{|})}{\Gamma(\beta)(x-y)^{\beta+\beta^{|}-1}}\ \{I_{0+}^{\alpha}\,t^{\beta-1}E_{\alpha,\beta}^{\gamma}(y+t)\}(x-y).$$

Further, by setting y= 0 in above equations, we get well- known result reported in [11] like as

$$M_{\alpha,\delta}^{\gamma}(\beta,\beta^{|};\mathrm{x},0)=\frac{\Gamma(\beta+\beta^{|})}{\Gamma(\gamma)\Gamma(\beta)}2\psi2\left[\begin{matrix}(\gamma,1), & (\beta,1) \\ (\beta+\beta^{|},1), & (\delta,1)\end{matrix} \; ,x\right].$$

In particular, when $\beta+\beta^{|}=\gamma$,

$$M_{\alpha,\delta}^{\gamma}(\beta,\gamma-\beta;\mathrm{x},0)=\frac{1}{\Gamma(\beta)}1\psi1\left[\begin{matrix}(\beta,1) \\ (\delta,\alpha)\end{matrix} \; ,x\right].$$

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