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RESEARCH ARTICLE

Bayes Estimator for the Scale Parameter of Laplace distribution under a Suggested Loss Function

Huda, A. Rasheed[1], Emad, F. AL-Shareefi[2]

1. AL-Mustansiriya University Collage of Science, Dept. of Math.

2. Southern Technical University/ Nasiriyah Technical Institute

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Corresponding Author

Huda,A.Rasheed

Abstract

In the present paper, we propose a new loss function, and obtained Bayesian estimation of the Scale parameter for the Laplace distribution under the proposed loss function with assuming that, the location parameter is unknown and estimated its by median.
We considered Non- informative prior for the Scale parameter using Jefferys prior information and Informative prior represented by Gamble Type II prior. All these estimators are compared empirically using Monte Carlo simulation. The performance of these estimators is compared depending on the mean square errors (MSE's).

INTRODUCTION

AL-Nasser and Saleh (2009) suggested the Generalized Square Error Loss function in estimating the scale parameter and the Reliability function for **Weibull** distribution, which introduced as follows[8],[6]

$$\mathbf{L}(\hat{\boldsymbol{\theta}}, \boldsymbol{\theta}) = \left(\sum_{j=0}^{k} \mathbf{a_j}\boldsymbol{\theta_j}\right)(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta})^2$$

Where a_j **is constant**, $\boldsymbol{\theta} > 0$, $j = 0,1,2,3,\dots k$

We have proposal a new loss function which is a Modified of the Generalized square error loss function as following

$\mathbf{L}^*(\hat{\boldsymbol{\theta}}, \boldsymbol{\theta}) = \frac{\left(\sum_{j=0}^{k} \mathbf{a_j}\boldsymbol{\theta_j}\right)(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta})^2}{\boldsymbol{\theta}^c}$; where, c is a constant

The probability density function of a Laplace distributed random variable is given by [7]

$$f(x|a, \theta) = \frac{1}{2\theta}\exp\left[-\frac{|x-a|}{\theta}\right] \qquad -\infty < x < \infty \qquad (1)$$

Where a ∈(−∞,∞) and θ > 0 are location and scale parameters, respectively.

The Cumulative distribution function is given by

$$F(x|\,a,\theta)=\begin{cases}1-\frac{1}{2}\exp\left[\frac{a-x}{\theta}\right] & \text{for } x\ \geq a\\ \frac{1}{2}\exp\left[\frac{x-a}{\theta}\right] & \text{for } x<a\end{cases}$$

With moment –generating function

$$M_x(t)=\frac{e^{t\mu}}{1-b^2t^2}$$

2. Posterior Distribution Using Jeffreys Prior Information **[3][5]**

Let us assume that, θ has Non-information prior density, defined as

$$g\propto\sqrt{I(\theta)} \qquad (2)$$

Where $I(\theta)$ represented Fisher information which defined as follows:

$$I(\theta)=-nE\left[\frac{\partial^2\ln f(x;a,\theta)}{\partial\theta^2}\right]$$

Hence,

$$g_1(\theta)=k\sqrt{-nE\left(\frac{\partial^2\ln f(x;a,\theta)}{\partial\theta^2}\right)} \qquad (3)$$

$$\ln f(x;a,\theta)=-\ln(2)-\ln(\theta)-\frac{|x-a|}{\theta}$$

$$\frac{\partial\ln f(x;a,\theta)}{\partial\theta}=-\frac{1}{\theta}+\frac{|x-a|}{\theta^2}$$

$$\frac{\partial^2\ln f(x;a,\theta)}{\partial\theta^2}=\frac{1}{\theta^2}-\frac{2\theta|x-a|}{\theta^4}$$

$$E\left[\frac{\partial^2\ln f(x;a,\theta)}{\partial\theta^2}\right]=\frac{1}{\theta^2}-\frac{2}{\theta^3}E[|x-a|]$$

Recall that, If $\times\sim\text{Laplace}(a,\theta)$, then $|x-a|\sim\text{Exponential}\left(\theta^{-1}\right)$, so

$$E\left[\frac{\partial^2\ln f(x;a,\theta)}{\partial\theta^2}\right]=\frac{-1}{\theta^2} \qquad (4)$$

After Substitution (4) into (3), we get

$$g_1=\frac{k}{\theta}\sqrt{n} \qquad (5)$$

Now, combining the prior (5) with the likelihood function, we have the posterior distribution of θ with Jeffrey's prior information given by $h(\theta|\underline{x})$:

$$h_1(\theta|\underline{x})=\frac{g(\theta)L(\theta;x_1x_2\dots\dots..x_n)}{\int_0^\infty g(\theta)L(\theta;x_1x_2\dots\dots..x_n)d\theta}$$

$$h_1(\theta|\underline{x})=\frac{\frac{1}{\theta^{n+1}}\exp\left[-\frac{\sum_{i=1}^n|x_i-a|}{\theta}\right]}{\int_0^\infty\frac{1}{\theta^{n+1}}\exp\left[-\frac{\sum_{i=1}^n|x_i-a|}{\theta}\right]d\theta} \qquad (6)$$

On simplification, we have

$$h_1(\theta|\underline{x})=\frac{\left(\sum_{i=1}^n|x_i-a|\right)^n e^{\frac{-\sum_{i=1}^n|x_i-a|}{\theta}}}{\theta^{n+1}\Gamma(n)}=\frac{T^n e^{\frac{-T}{\theta}}}{\theta^{n+1}\Gamma(n)} \qquad (7)$$

$$\text{Where } T=\sum_{i=1}^n|x_i-a|$$

Note that, this posterior density is recognized as the density of the Inverse Gamma (IG) distribution: $\theta|\underline{x}\sim IG(n,T)$

3.Posterior Distribution Using Gumbel Type-II prior**[2]:**

The prior predictive prior distribution using Gambel Type IIdistribution is defined as follow

$g_2(\theta) = b\left(\frac{1}{\theta}\right)^2 \exp\left[\frac{-b}{\theta}\right] \quad b, \theta > 0$ (8)

Combining the prior with the likelihood and using Bayes theorem we have updated information about the parameter of interest θ represented by the posterior distribution of θ with density $h_2(\boldsymbol{\theta}|\underline{x})$.

Now, combining the prior (8)with the likelihood function(1),we have the posterior distribution of θ with Gambel Type II prior distribution denoted by $h_2(\boldsymbol{\theta}|\underline{x})$

$$\mathbf{h_2}(\boldsymbol{\theta}|\underline{\mathbf{x}}) = \frac{\mathbf{g_2}(\boldsymbol{\theta})\mathbf{L}(\boldsymbol{\theta}; \mathbf{x_1 x_2} \dots \dots . . \mathbf{x_n})}{\int_0^\infty \mathbf{g_2}(\boldsymbol{\theta})\mathbf{L}(\boldsymbol{\theta}; \mathbf{x_1 x_2} \dots \dots . . \mathbf{x_n})\mathbf{d}\boldsymbol{\theta}} \quad (9)$$

$$\mathbf{h_2}(\boldsymbol{\theta}|\underline{\mathbf{x}}) = \frac{\frac{1}{\boldsymbol{\theta}^{\mathbf{n+2}}} \mathbf{e}^{-(\sum_{i=1}^{n}|\mathbf{x_i}-\mathbf{a}|+\mathbf{b})/\boldsymbol{\theta}}}{\int_0^\infty \frac{1}{\boldsymbol{\theta}^{\mathbf{n+2}}} \mathbf{e}^{-(\sum_{i=1}^{n}|\mathbf{x_i}-\mathbf{a}|+\mathbf{b})/\boldsymbol{\theta}} \mathbf{d}\boldsymbol{\theta}} = \frac{\frac{1}{\boldsymbol{\theta}^{\mathbf{n+2}}} \mathbf{e}^{-\mathbf{P}/\boldsymbol{\theta}}}{\int_0^\infty \frac{1}{\boldsymbol{\theta}^{\mathbf{n+2}}} \mathbf{e}^{-\mathbf{P}/\boldsymbol{\theta}} \mathbf{d}\boldsymbol{\theta}}$$

Where P = $\sum_{i=1}^{n}|x_i - a| + b$

$$h_2(\boldsymbol{\theta}|\underline{x}) = \frac{\frac{1}{\boldsymbol{\theta}^{\mathbf{n+2}}} \mathbf{e}^{-\mathbf{P}/\boldsymbol{\theta}}}{\int_0^\infty \frac{1}{\boldsymbol{\theta}^{\mathbf{n+2}}} \mathbf{e}^{-\mathbf{P}/\boldsymbol{\theta}} \mathbf{d}\boldsymbol{\theta}} = \frac{\frac{1}{\boldsymbol{\theta}^{\mathbf{n+2}}} \mathbf{e}^{-\mathbf{P}/\boldsymbol{\theta}}}{\frac{\Gamma(n+\mathbf{1})}{\mathbf{P}^{\mathbf{n+1}}}}$$

$$h_2(\boldsymbol{\theta}|\underline{x}) = \frac{(\sum_{i=1}^{n}|x_i - a| + b)^{\mathbf{n+1}} \mathbf{e}^{-(\sum_{i=1}^{n}|x_i - a| + b)/\boldsymbol{\theta}}}{\boldsymbol{\theta}^{\mathbf{n+2}}\Gamma(n+\mathbf{1})} \quad (10)$$

This implies that,$(\boldsymbol{\theta}|\underline{\mathbf{x}}) \sim \mathbf{Inv-Gamma}(\mathbf{n+1}, (\sum_{i=1}^{n}|x_i - a| + b))$

Such that, $\mathbf{E}(\boldsymbol{\theta}) = \frac{\sum_{i=1}^{n}|x_i - a| + b}{\mathbf{n}}$, $\mathbf{V}(\boldsymbol{\theta}) = \frac{(\sum_{i=1}^{n}|x_i - a| + b)^{\mathbf{2}}}{\mathbf{n^2(n-1)}}$,

4. Bayesian Estimators under Modified Generalized Square Error Loss Function

We have suggested anew loss function is called Modified Generalized Square Error Loss function (MGSELF) as fallowing:

$$\mathbf{L}^*(\hat{\boldsymbol{\theta}}, \boldsymbol{\theta}) = \frac{(\sum_{j=0}^{k} \mathbf{a_j}\boldsymbol{\theta}_\mathbf{j})(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta})^{\mathbf{2}}}{\boldsymbol{\theta}^{\mathbf{c}}}$$

Then, the Risk function under the Modify Generalized Square Error Loss function **(MGSELF)** denoted by $R_{MGS}(\hat{\theta}, \theta)$, will be

$R_{MGS}(\hat{\theta}, \theta) = E[L_{MGS}(\hat{\theta}, \theta)]$

$$\mathbf{R_{MGS}}(\hat{\boldsymbol{\theta}}, \boldsymbol{\theta}) = \int_0^\infty \frac{\mathbf{1}}{\boldsymbol{\theta}^{\mathbf{C}}}\left(\sum_{j=0}^{k} \mathbf{a_j}\boldsymbol{\theta}_\mathbf{j}\right)(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta})^{\mathbf{2}}\mathbf{h}(\boldsymbol{\theta}|\underline{\mathbf{x}})\mathbf{d}\boldsymbol{\theta}$$

$$= \int_0^\infty \frac{1}{\theta^C}(a_0 + a_1\theta + \dots + a_k\theta^k)(\hat{\boldsymbol{\theta}}^{\mathbf{2}} - \mathbf{2}\hat{\boldsymbol{\theta}}\boldsymbol{\theta} + \boldsymbol{\theta}^{\mathbf{2}})\mathbf{h}(\boldsymbol{\theta}|\underline{x})d\boldsymbol{\theta}$$

$$=a_0\hat{\boldsymbol{\theta}}^{\mathbf{2}}E\left(\frac{1}{\theta^C}|\underline{x}\right) - 2a_0\hat{\boldsymbol{\theta}}E\left(\frac{1}{\theta^{C-1}}|\underline{x}\right) + a_0E\left(\frac{1}{\theta^{C-2}}|\underline{x}\right) + a_1\hat{\boldsymbol{\theta}}^{\mathbf{2}}E\left(\frac{1}{\theta^{C-1}}|\underline{x}\right) - 2a_1\hat{\boldsymbol{\theta}}E\left(\frac{1}{\theta^{C-2}}|\underline{x}\right) + a_1E\left(\frac{1}{\theta^{C-3}}|\underline{x}\right) + \dots +$$
$$a_K\hat{\boldsymbol{\theta}}^{\mathbf{2}}E\left(\frac{1}{\theta^{C-K}}|\underline{x}\right) - 2a_K\hat{\boldsymbol{\theta}}E\left(\frac{1}{\theta^{C-(K+1)}}|\underline{x}\right) + a_KE\left(\frac{1}{\theta^{C-(K+2)}}|\underline{x}\right)$$

Taking the partial derivative for $R_{MGS}(\hat{\theta}, \theta)$ with respect to $\hat{\boldsymbol{\theta}}$ and setting it equal to zero, yields

$$\frac{R_{MGS}(\hat{\theta}, \theta)}{\partial\hat{\theta}} = 2a_0\hat{\boldsymbol{\theta}}\,E\left(\frac{1}{\theta^C}|\underline{x}\right) - 2a_0\,E\left(\frac{1}{\theta^{C-1}}|\underline{x}\right) + 2a_1\hat{\boldsymbol{\theta}}E\left(\frac{1}{\theta^{C-1}}|\underline{x}\right) - 2a_1E\left(\frac{1}{\theta^{C-2}}|\underline{x}\right) + \dots + 2a_K\hat{\boldsymbol{\theta}}E\left(\frac{1}{\theta^{C-K}}|\underline{x}\right) - 2a_KE\left(\frac{1}{\theta^{C-(K+1)}}|\underline{x}\right) = 0$$

$$\hat{\boldsymbol{\theta}}_{MGS} = \frac{a_0E\left(\frac{1}{\theta^{C-1}}|\underline{x}\right) + a_1E\left(\frac{1}{\theta^{C-2}}|\underline{x}\right) + \dots + a_KE\left(\frac{1}{\theta^{C-(K+1)}}|\underline{x}\right)}{a_0E\left(\frac{1}{\theta^C}|\underline{x}\right) + a_1E\left(\frac{1}{\theta^{C-1}}|\underline{x}\right) + \dots + a_KE\left(\frac{1}{\theta^{C-K}}|\underline{x}\right)} \quad (11)$$

Lemma 1: **The mth moment for $\boldsymbol{\theta}$ to** $h_1(\boldsymbol{\theta}|\underline{x})$**is**

$$\mathbf{E}(\boldsymbol{\theta}^{\mathbf{m}}) = \int_0^\infty \boldsymbol{\theta}^{\mathbf{m}}\,\mathbf{h_1}(\boldsymbol{\theta}|\underline{\mathbf{x}})\mathbf{d}\boldsymbol{\theta}$$
$$= \frac{(\sum_{i=1}^{n}|x_i - a|)^{\mathbf{m}}\Gamma(n-m)}{\Gamma(n)} , \quad \mathbf{m} = \mathbf{0}, \mathbf{1}, \dots$$

Lemma 2:The mth moment for $\boldsymbol{\theta}$ to $h_2(\boldsymbol{\theta}|\underline{x})$**is**

$$\mathbf{E}(\boldsymbol{\theta}^{\mathbf{m}}) = \int_{0}^{\infty} \boldsymbol{\theta}^{\mathbf{m}}\, \mathbf{h_2}(\boldsymbol{\theta}|\underline{\mathbf{x}})\mathbf{d\theta}$$

$$= \frac{(\sum_{i=1}^{n}|x_i - a|)^{\mathbf{m}}\Gamma(n-m)}{\Gamma(n)} \quad , \mathbf{m} = \mathbf{0}, \mathbf{1}, \ldots$$

Lemma 3:The m^th^ moment for $\frac{1}{\theta^m}$ to $h_1(\boldsymbol{\theta}|\underline{x})$ is

$$\mathbf{E}\left(\frac{\mathbf{1}}{\boldsymbol{\theta}^{\mathbf{m}}}\right) = \int_{0}^{\infty} \frac{\mathbf{1}}{\boldsymbol{\theta}^{\mathbf{m}}} h_1(\boldsymbol{\theta}|\underline{x})d\theta$$

$$= \frac{\Gamma(n+m)}{\Gamma(n)\sum_{i=1}^{n}|x_i - a|^m}, \qquad \mathbf{m} = \mathbf{0}, \mathbf{1}, \ldots$$

Lemma 4:The m^th^ moment for $\frac{1}{\theta^m}$ to $h_2(\boldsymbol{\theta}|\underline{x})$ is

$$\mathbf{E}\left(\frac{\mathbf{1}}{\boldsymbol{\theta}^{\mathbf{m}}}\right) = \int_{0}^{\infty} \frac{\mathbf{1}}{\boldsymbol{\theta}^{\mathbf{m}}} h_2(\boldsymbol{\theta}|\underline{x})d\theta$$

$$= \frac{\Gamma(n+m+1)}{\Gamma(n+1)(\sum_{i=1}^{n}|x_i - a| + b)^m} \quad , \qquad \mathbf{m} = \mathbf{0}, \mathbf{1}, \ldots$$

4.1 Bayes estimator with Jeffreys prior information

1- If K=1 and C=1, then

$$\hat{\boldsymbol{\theta}}_{MGSJ\ 1} = \frac{a_0(n-1)\sum_{i=1}^{n}|x_i-a| + a_1\sum_{i=1}^{n}|x_i-a|^2}{a_0 n(n-1) + a_1(n-1)\sum_{i=1}^{n}|x_i-a|} \qquad (11)$$

2- If K=2 and C=1, then

$$\hat{\boldsymbol{\theta}}_{MGSJ\ 2} = \frac{a_0(n-1)(n-2)\sum_{i=1}^{n}|x_i-a| + a_1\sum_{i=1}^{n}|x_i-a|^2 + a_2\sum_{i=1}^{n}|x_i-a|^3}{a_0 n(n-1)(n-2) + a_1(n-1)(n-2)\sum_{i=1}^{n}|x_i-a| + a_2(n-2)\sum_{i=1}^{n}|x_i-a|^2} \qquad (12)$$

3- If K=1 and C=2, then,

$$\hat{\boldsymbol{\theta}}_{MGSJ\ 3} = \frac{a_0 n\sum_{i=1}^{n}|x_i-a| + a_1\sum_{i=1}^{n}|x_i-a|}{a_0 n(n+1) + a_1 n\sum_{i=1}^{n}|x_i-a|} \qquad (13)$$

4- If K=2 and C=2, then,

$$\hat{\boldsymbol{\theta}}_{MGSJ\ 4} = \frac{a_0 n(n-1)\sum_{i=1}^{n}|x_i-a| + a_1(n-1)\sum_{i=1}^{n}|x_i-a|^2 + a_2\sum_{i=1}^{n}|x_i-a|^3}{a_0 n(n+1)(n-1) + a_1 n(n-1)\sum_{i=1}^{n}|x_i-a| + a_2(n-1)\sum_{i=1}^{n}|x_i-a|} \qquad (14)$$

5- If K=1 and C=3, than,

$$\hat{\boldsymbol{\theta}}_{MGSJ\ 5} = \frac{a_0 n(n+1)\sum_{i=1}^{n}|x_i-a| + a_1 n\sum_{i=1}^{n}|x_i-a|^2}{a_0 n(n+1)(n+2) + a_1 n(n+1)} \qquad (15)$$

6- If K=2 and C=3, then

$$\hat{\boldsymbol{\theta}}_{MGSJ\ 6} = \frac{a_0 n(n+1)\sum_{i=1}^{n}|x_i-a| + a_1 n\sum_{i=1}^{n}|x_i-a|^2 + a_2\sum_{i=1}^{n}|x_i-a|^3}{a_0 n(n+1)(n+1) + a_1 n(n+1)\sum_{i=1}^{n}|x_i-a| + a_2 n\sum_{i=1}^{n}|x_i-a|^2} \qquad (16)$$

4.2 Bayes estimator with Gamble Type II prior information

1-If K=1 and C=1, then

$$\hat{\boldsymbol{\theta}}_{MGSG\ 1} = \frac{a_0 n(\sum_{i=1}^{n}|x_i-a|+b) + a_1(\sum_{i=1}^{n}|x_i-a|+b)^2}{a_0 n(n+1) + a_1 n(\sum_{i=1}^{n}|x_i-a|+b)} \qquad (17)$$

2- If K=2 and C=1, then

$$\hat{\boldsymbol{\theta}}_{MGSG\ 2} = \frac{a_0 n(n-1)(\sum_{i=1}^{n}|x_i-a|+b) + a_1(n-1)(\sum_{i=1}^{n}|x_i-a|+b)^2 + a_2(\sum_{i=1}^{n}|x_i-a|+b)^3}{a_0 n(n+1)(n-1) + a_1 n(n-1)(\sum_{i=1}^{n}|x_i-a|+b) + a_2(n-1)(\sum_{i=1}^{n}|x_i-a|+b)^2} \qquad (18)$$

3- If K=1 and C=2, then

$$\hat{\boldsymbol{\theta}}_{MGSG\ 3} = \frac{a_0(n+1)(\sum_{i=1}^{n}|x_i-a|+b) + a_1(\sum_{i=1}^{n}|x_i-a|+b)^2}{a_0(n+1)(n+2) + a_1(n+1)(\sum_{i=1}^{n}|x_i-a|+b)} \qquad (19)$$

4- **If K=2 and C=2,** then

$$\hat{\boldsymbol{\theta}}_{MGSG\ 4}=\frac{a_0 n(n+1)\left(\sum_{i=1}^{n}|x_i-a|+b\right)+a_1 n\left(\sum_{i=1}^{n}|x_i-a|+b\right)^2+a_2\left(\sum_{i=1}^{n}|x_i-a|+b\right)^3}{a_0(n+1)(n+2)+a_1 n(n+1)\left(\sum_{i=1}^{n}|x_i-a|+b\right)+a_2 n\left(\sum_{i=1}^{n}|x_i-a|+b\right)^2} \quad \textbf{(20)}$$

5- If K=1 and C= 3, then

$$\hat{\boldsymbol{\theta}}_{MGSG\ 5}=\frac{a_0(n+1)(n+2)\left(\sum_{i=1}^{n}|x_i-a|+b\right)+a_1(n+1)\left(\sum_{i=1}^{n}|x_i-a|+b\right)^2}{a_0(n+1)(n+2)(n+3)+a_1(n+1)(n+2)\left(\sum_{i=1}^{n}|x_i-a|+b\right)} \quad (21)$$

6- If K=2 and C=3, then,

$$\hat{\boldsymbol{\theta}}_{MGSG\ 6}=\frac{a_0 n(n+1)(n+2)\left(\sum_{i=1}^{n}|x_i-a|+b\right)+a_1(n+1)\left(\sum_{i=1}^{n}|x_i-a|+b\right)^2+a_2\left(\sum_{i=1}^{n}|x_i-a|+b\right)^3}{a_0(n+1)(n+2)(n+3)+a_1(n+1)(n+2)\left(\sum_{i=1}^{n}|x_i-a|+b\right)+a_2(n+1)\left(\sum_{i=1}^{n}|x_i-a|+b\right)^2} \quad (22)$$

5. Maximum Likelihood Estimator of Location parameter

Let $x_1, x_2, \ldots, x_n$, be n independent and identically distributed samples the Maximum likelihood estimator $\hat{a}$of a is the sample median[16], In a special case of the Laplace distribution given as following:

$$f(x;a,\theta)=\frac{\sqrt{2}}{2\theta}e^{-\frac{\sqrt{2}|x-a|}{\theta}} \qquad ,x\in R$$

$$f(x;a,\theta)=\frac{1}{\sqrt{2}\theta}e^{-\frac{\sqrt{2}|x-a|}{\theta}}$$

Consider the likelihood function for n data samples [9]

$$L(a,\theta;x)=\prod_{i=1}^{n}\frac{1}{\sqrt{2}\theta}e^{-\frac{\sqrt{2}|x-a|}{\theta}}$$

$$=\left(\sqrt{2}\theta\right)^{-n}e^{\frac{-\sqrt{2}}{\theta}\sum_{i=1}^{n}|x_i-a|}$$

Take the log Likelihood function as $L(a,\theta;x)=Log\left(L(a,\theta;x)\right)$ and we get

$$L(a,\theta;x)=-n\ln\left(\sqrt{2}\theta\right)-\frac{\sqrt{2}}{\theta}\sum_{i=1}^{n}|x_i-a|$$

Take the partial derivative with respect to the parameter a

$$\frac{\partial L}{\partial a}=\frac{\sqrt{2}}{\theta}\sum_{i=1}^{n}\frac{\partial|x_i-a|}{\partial a}$$

Which is equal to $=\frac{\sqrt{2}}{\theta}\sum_{i=1}^{n}\operatorname{sgn}|x_i-a|$

Using the identity

$$\frac{\partial|x|}{\partial x}=\frac{\partial\sqrt{x^2}}{\partial x}=x(x^2)^{-\frac{1}{2}}=\frac{x}{|x|}\operatorname{sgn}(n)$$

To maximize the likelihood function we need to solve

$$\frac{\sqrt{2}}{\theta}\sum_{i=1}^{n}\operatorname{sgn}|x_i-a|=0$$

In summary, $\hat{a}=\text{median }(x_1,\ldots x_n)$, is the maximum likelihood estimator for any n.

6. Simulation Results

In our simulation study, we generated I = 5000 samples of size n = 10, 50 and 100 from Laplace distribution to represent small, moderate and large sample size with the scale parameter θ =0.5, 1.5, and 3, with location parameter, $a = 1.5$, and the constants, $a_0 = 1.5$, $a_1 = 1$, $a_2 = 0.5$.

In this section, Monte-Carlo simulation study is performed to compare the methods of estimation by using mean square Errors (MSE's) as an index for precision to compare the efficiency of each of estimators, where:

$$MSE(\hat{\theta})=\frac{\sum_{i=1}^{I}\left(\hat{\theta}_i-\theta\right)^2}{I}$$

We use QBasic program to find the values of MSE. The results were summarized and tabulated in the following tables and figures as following:

Table (1): The Expected values and (MSE) of the Different Estimators for Laplace Distribution where $\theta = 0.5$ and $b = 1.2$

Estimator	Criteria \ n	10	50	100
MGSJ1	EXP	0.48623	0.49727	0.49912
	MSE	0.02615	0.00505	0.00250
MGSJ2	EXP	0.49366	0.49838	0.49965
	MSE	0.02795	0.00510	0.00251
MGSJ3	EXP	0.35008	0.37002	0.37278
	MSE	0.03020	0.01847	0.01697
MGSJ4	EXP	0.46927	0.51811	0.52518
	MSE	0.03085	0.00701	0.00401
MGSJ5	EXP	0.49163	0.62465	0.64588
	MSE	0.03942	0.02786	0.02779
MGSJ6	EXP	0.33374	0.37219	0.37753
	MSE	0.03597	0.01817	0.01592
MGSG1	EXP	0.55374	0.51117	0.50609
	MSE	0.02428	0.00497	0.00249
MGSG2	EXP	0.56226	0.51233	0.50663
	MSE	0.02691	0.00505	0.02513
MGSG3	EXP	0.50562	0.50126	0.50110
	MSE	0.01779	0.04666	0.00240
MGSG4	EXP	0.51174	0.50233	0.50163
	MSE	0.01898	0.00472	0.00242
MGSG5	EXP	0.46527	0.49172	0.49622
	MSE	0.016185**	0.00455**	0.00237**
MGSG6	EXP	0.46980	0.49272	0.49673
	MSE	0.01663	0.00458	0.00238
Best Estimator		MGSG5	MGSG5	MGSG5

The results in table (1) show that, (MGSG5) estimator is the best for all sample sizes, followed by (MGSG6) estimator, while (MGSJ1) is the best estimator with Jeffreys prior.

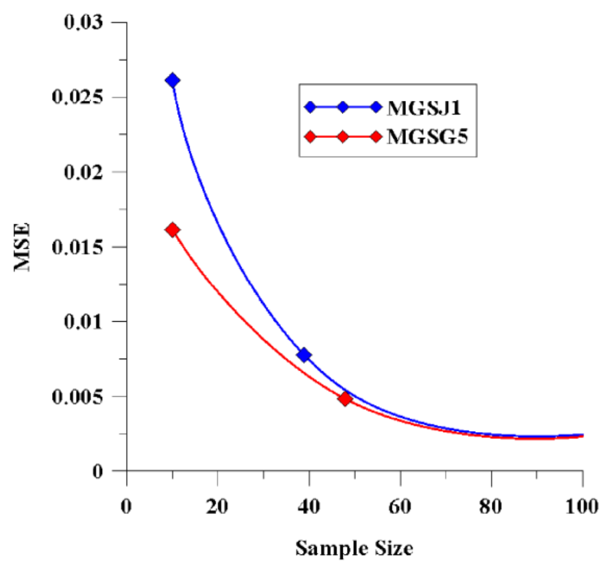

Fig.1: MSE's of two estimators of the scale parameter of Laplacedistribution with $\theta = 0.5$ and $b = 1.2$

Table (2): The Expected values and (MSE) of the Different Estimators for Laplace Distribution where $\theta = 1.5$ $and b = 3.6$

Estimator	n / Criteria	10	50	100
MGSJ1	**EXP**	1.49739	1.49938	1.50112
	MSE	0.24895	0.04601	0.02269
MGSJ2	**EXP**	1.57917	1.51232	1.50744
	MSE	0.30962	0.04768	0.02311
MGSJ3	**EXP**	0.72079	0.74468	0.74774
	MSE	0.62365	0.57341	0.56731
MGSJ4	**EXP**	1.92144	2.03270	2.05021
	MSE	1.03327	0.45341	0.38704
MGSJ5	**EXP**	2.18367	2.80372	2.90338
	MSE	1.59603	2.05787	2.15907
MGSJ6	**EXP**	0.87109	0.92827	0.94729
	MSE	0.47123	0.33171	0.31359
MGSG1	**EXP**	1.70214	1.54118	1.52206
	MSE	0.24472	0.04590	0.02272
MGSG2	**EXP**	1.79177	1.55452	1.52848
	MSE	0.32911	0.04863	0.03241
MGSG3	**EXP**	1.55032	1.51110	1.50703
	MSE	0.17100	0.01523	0.02185
MGSG4	**EXP**	1.61845	1.52370	1.51326
	MSE	0.21090	0.04441	0.02232
MGSG5	**EXP**	1.42358	1.48218	1.49229
	MSE	**0.14742****	**0.04119****	**0.02143****
MGSG6	**EXP**	1.47642	1.49408	1.49834
	MSE	0.16277	0.04217	0.02171
Best Estimator		MGSG5	MGSG5	MGSG5

The results in table (2) show that, (MGSG5) estimator is the best for all sample sizes, followed by (MGSG6) estimator, and (MGSJ1) is the best estimator with Jeffreys prior.

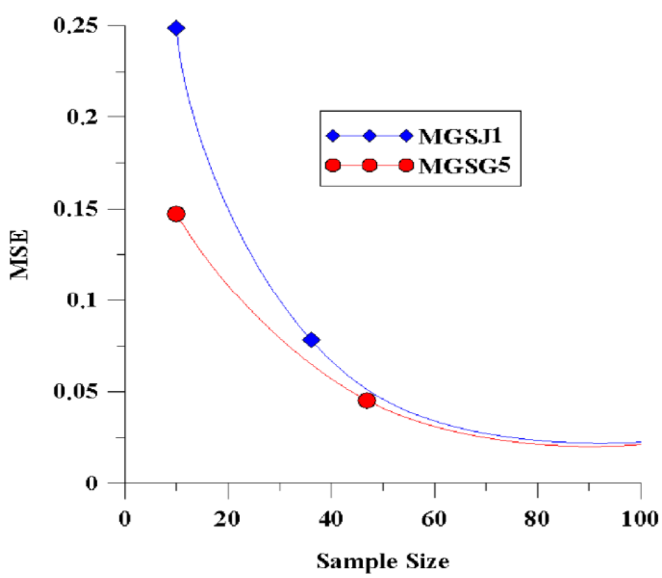

Fig.2: MSE's of two estimators of the scale parameter of Laplacedistribution with $\theta = 1.5$ and $b = 3.6$

Table (3): The Expected values and (MSE) of the Different Estimators for Laplace Distribution where $\theta = 3\ and b = 7.3$

Estimator	Criteria \ n	10	50	100
MGSJ1	**EXP**	3.04621	3.00882	3.00727
	MSE	1.02636	0.18514	0.09069
MGSJ2	**EXP**	3.30524	3.05079	3.02778
	MSE	1.38541	0.19541	0.09365
MGSJ3	**EXP**	0.98696	0.99846	0.99963
	MSE	4.06795	4.00856	4.00264
MGSJ4	**EXP**	5.83026	5.96339	5.99086
	MSE	18.9297	10.8217	9.95104
MGSJ5	**EXP**	6.49868	8.39674	8.70396
	MSE	24.6833	33.0997	34.6400
MGSJ6	**EXP**	1.83416	1.96684	1.98561
	MSE	1.84149	1.17396	1.08265
MGSG1	**EXP**	3.46627	3.09444	3.05015
	MSE	1.05212	0.18670	0.09172
MGSG2	**EXP**	3.74196	3.13726	3.07088
	MSE	1.58272	0.20397	0.09604
MGSG3	**EXP**	3.15376	3.03389	3.01999
	MSE	0.71312	0.17202	0.08785
MGSG4	**EXP**	3.37059	3.07465	3.04020
	MSE	0.97130	0.18334	0.09082
MGSG5	**EXP**	2.89320	2.97567	2.99041
	MSE	**0.5905****	**0.16495****	**0.08583****
MGSG6	**EXP**	3.06686	3.01444	3.01014
	MSE	0.69203	0.17105	0.08755
Best Estimator		MGSG5	MGSG5	MGSG5

The results in table (3) show that, (MGSG5) estimator is the best for all sample sizes, followed by ((MGSG6)) estimator, while (MGSJ1) is the best estimator with Jeffreys prior.

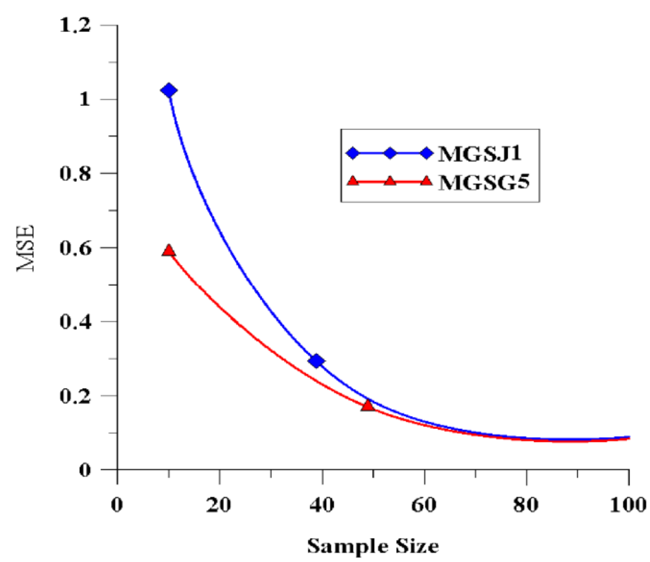

Fig.3: MSE's of two estimators of the scale parameter of Laplacedistribution with $\theta = 3\ and\ b = 7.3$

7. Discussion

The results of the simulation study for estimating the scale parameter $(\theta)$of Laplace distribution when the location parameter (α) isunknown and estimated by median, are summarized and tabulated in tables (1, 2 and 3) which contain the expected values of the scale parameter θ and MSE's, we have observed that:

- The performance of Bayes estimator under Modify Generalized Square Error Loss function (MGSELF) with Gamble Type II prior information is the best estimator, comparing with Jeffery prior information for all sample sizes and with all values of the scale parameter.
- The best estimator for the scale parameter of Laplace distribution is (MGSG5)which represented the estimator withGamble Type II prior information when K=1 and C= 3, for all sample sizes and with all values of the scale parameter, followed by (MGSG6) which represented the estimator with Gamble Type II prior information when K=2 and C= 3.
- The performance of (MGSJ1) is the best estimator with Jeffery prior information to other estimator for (10, 50 and 100) sample sizes and with all values of the scale parameter.

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